Entropy, Markov Information Sources and Parrondo Games

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Abstract. Recently Parrondo's paradoxical games have been connected with the Brownian ratchet. We consider some open questions raised in that context.

INTRODUCTION

Noise is associated with unpredictability and disorder, which are measured by entropy. A main line of development can be traced from the work of Clausius and Boltzmann in physics last century through Shannon's work fifty years ago on what is now information theory to the ideas of Kolmogorov in 1955 on entropy as a measure of capacity in metric spaces. The area is rich in surprises and unexpected connections. An example of the former is a result of Plaskota [1] that noisy information can sometimes be better than exact information. An example of the latter is provided by an invited paper at this conference [2], which draws together the Brownian ratchet and Parrondo games that can be represented as skip–free Markov chains.

The latter paper raises a number of open questions. In particular, suppose we associate such a chain with a sequence of bits, a one being produced whenever an rightward step is taken and a zero for each leftward step. We may further associate an entropy with such a sequence. How does this entropy relate to the underlying game? We present some preliminary ideas from an information–theoretic standpoint on the open questions raised in [2].

PARRONDO GAMES AND ENTROPY

The central example in [2] involves two games, which can be represented as skip–free discrete–time Markov chains on the integers, each with a leftward drift. In game A, the player makes a sequence of steps, each being either rightward (with

1) Acknowledgement. This work was supported financially by DSTO.

CP511, *Unsolved Problems of Noise and Fluctuations,* edited by D. Abbott and L. B. Kish
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probability p) or leftward (with probability $1-p$). Game B is similar except that the probabilities are state–dependent and periodic, the rightward ones being being p_1 for states of the form $3i$

and p_2 for all other states. A mixed game C is formed with p_i replaced by

$$q_i = \gamma p + (1-\gamma)p_i, \quad (i = 1, 2) \tag{1}$$

with γ ($0 < \gamma < 1$) representing a mixing probability. It is shown in [2] that passage to the origin from an initial positive state is certain for games A and B under the respective conditions

$$(1-p)/p > 1, \quad (1-p_1)(1-p_2)^2/(p_1 p_2)^2 > 1, \tag{2}$$

while game C will escape to infinity (without visiting the origin) with positive probability if

$$(1-q_1)(1-q_2)^2/(q_1 q_2^2) < 1. \tag{3}$$

If (2) and (3) hold, then the mixed chain drifts to the right while the component chains drift to the left. Remarkably, this is the case for $p = 5/11$, $p_1 = 1/121$, $p_2 = 10/11$ and $\gamma = 1/2$.

A game is *fair* if there is no drift. This occurs if the corresponding inequality from (2) and (3) instead occurs as an equality. When (3) holds we refer to game C as *winning*.

Harmer *et al.* raise a number of questions including the following. Each chain can be associated with the production of a sequence of bits, zeros corresponding to downward steps and ones to upward. Game A is then associated with a corresponding entropy $S_A := -p \log p - (1-p)\log(1-p)$. Correspondingly S_B, S_C are entropies associated with games B and C. The following questions are proposed.
Question 1. Are the entropies related to the fairness of the games?
Question 2. What is the relationship between S_C and S_A and S_B and how can S_C be smaller than S_A?

As noted in [2], the definitions of S_B, S_C are more complicated because of successive steps being correlated. Indeed, we have a further complication in that entropy is usually defined for a Markov chain only if the chain is irreducible and positive recurrent, which is not the situation here.

We propose the following, which is consistent with the above definition of S_A and easily extended in an obvious way whenever the transition probabilities are periodic functions of state. Associate with A a Markov chain A' with state space $\{1, 2, 3\}$, A' being in state $j \in \{1, 2, 3\}$ whenever A is in a state congruent to j modulo 3. (For the version of the game on the nonnegative integers with zero as an absorbing state, we regard A as restarting in 3 whenever it enters state zero.) The same device provides associated Markov chains B', C' corresponding to B and C. The derived chains then have one–step transition matrices of the form

$$P = \begin{pmatrix} 0 & r_1 & 1-r_1 \\ 1-r_2 & 0 & r_2 \\ r_3 & 1-r_3 & 0 \end{pmatrix}, \tag{4}$$

with $r_i \equiv p$ for A', $r_1 = r_2 = p_2$ and $r_3 = p_1$ for B' and $r_1 = r_2 = q_2$ and $r_3 = q_1$ for C'.

These three new chains are all positive recurrent and irreducible and generate bits 1 corresponding to transitions $i \to i+1$ (mod 3) and 0 corresponding to transitions $i \to i-1$ (mod 3). The chains correspond to unifilar sources (see [3, Section 6.5]). The chain A' has equilibrium probabilities $\pi_i^A = 1/3$ $(i = 1, 2, 3)$ and associated entropy S_A.

The results for B' and C' are more complicated. We may solve the global balance equations $\pi_i^C = \sum_{j=1}^3 \pi_j^C P_{j,i}^C$ $(i = 1, 2, 3)$ for chain C' with the normalizing condition $\sum_{i=1}^3 \pi_i^C = 1$ to obtain the equilibrium probabilities

$$\pi_1^C = \frac{1 - q_2 + q_1 q_2}{D^C}, \quad \pi_2^C = \frac{1 - q_1 + q_1 q_2}{D^C}, \quad \pi_3^C = \frac{1 - q_2 + q_2^2}{D^C}, \tag{5}$$

where $D^C := 3 - q_1 - 2q_2 + 2q_1 q_2 + q_2^2$. The ergodic probabilities for B' are given by the same expressions with p_1, p_2 replacing respectively q_1 and q_2. We refer to the version of (5) for B as (5′).

Since these chains constitute unifilar Markov sources, the corresponding entropies are

$$S_B = \left[\pi_1^B + \pi_2^B\right] J(p_2) + \pi_3^B J(p_1), \quad S_C = \left[\pi_1^C + \pi_2^C\right] J(q_2) + \pi_3^C J(q_1), \tag{6}$$

where $J(x) := -x \log x - (1-x)\log(1-x)$ $(0 < x < 1)$. Because $J(x)$ is strictly concave and is symmetric about $x = 1/2$, it is strictly monotone increasing on $[0, 1/2]$ and we have that $J(x) > J(y)$ whenever $|x - 1/2| < |y - 1/2|$ and that $J(x)$ takes its maximum at $x = 1/2$. We are now in a position to address the proposed questions.

Question 2

We have $S_A = J(p)$, which with (6) indicates a quite complicated relation between S_A, S_B and S_C. Turning to the latter part of the question, we shall show first that $S_A > S_C$ for a range of parameter values subsuming those of [2]. A comprehensive numerical analysis is beyond our page limits, but some indicative results can be given. To begin we remark that $p_2 > p_1$ suffices to give

$$1 - p_i + p_i p_2 > 1 - p_2 + p_i p_2 \quad (i = 1, 2),$$

so (5′) yields $\pi_2^B > \pi_3^B$ and $\pi_3^B > \pi_1^B$. Hence $\pi_3^B < 1/2$. Also, $p_2 > p_1$ implies $q_2 > q_1$, so we have also $\pi_2^C > \pi_3^C > \pi_1^C$.

Suppose game C is winning and $p = 5/11$. For $0 < \gamma < 1$, having $1/2 > p > p_1$ provides

$$J(p) > J(q_1) > J(p_1). \tag{7}$$

Since $q_1 < 5/11$, we have $(1-q_1)/q_1 > 6/5$, so from (3) game C can be winning only if $(1-q_2)/q_2 < \sqrt{5/6}$, that is, if $q_2 \geq 1/(1+\sqrt{5/6})$. Thus game C can be winning only for $\gamma \leq \gamma_0 := [10 - 11/(1+\sqrt{5/6})]/5 \approx 0.849896$. When this holds, $q_1 \leq q := (5/11)\gamma_0 + (1-\gamma_0)/121 \approx 0.386316$ and again for (3) to hold we require $(1-q_2)^2/q_2^2 < q/(1-q)$, or $q_2 > (1-q)^{1/2})/[q^{1/2} + (1-q)^{1/2}]$ (≈ 0.55695) $> 6/11$.

Thus $6/11 < q_2 \leq p_2$, so that

$$J(p) > J(q_2) \geq J(p_2). \tag{8}$$

By (6), S_C is a convex linear combination of $J(q_1)$ and $J(q_2)$ and so by (7), (8) $S_A = J(p) > S_C$ whenever C is winning.

In fact it is also the case that $S_B < S_A$ whenever $1 > p_2 \geq 1/2$ and $p_1 \leq 1/6$. For $p_2 > p_1$ and $p_2 \geq 1/2$ gives $2p_2(p_2 - p_1) > p_2 - p_1$ and so

$$(1 - p_2 + p_1 p_2) + (1 - p_1 + p_1 p_2) < 2(1 - p_2 + p_2^2).$$

Hence $\pi_1^B + \pi_2^B < 2\pi_3^B$ and since $\sum_{i=1}^3 \pi_i^B = 1$, we can deduce that $\pi_3^B > 1/3$ and thus $\pi_1^B + \pi_2^B < 2/3$. Therefore from (6) $S_B < (2/3)J(1/2) + (1/2)J(p_1)$. It suffices for $S_B < S_A = J(p)$ that $J(p_1) \leq 2\,[J(5/11) - (2/3)J(1/2)] \approx 0.654725$ (taking logs to base 2). The desired result follows from the monotonicity of $J(x)$ on $[0, 1/2]$, since $J(1/6) \approx 0.650022$.

Question 1

As a preliminary we note that for a matrix of the form (4) the entropy is maximized when $r_1 = r_2 = r_3 = 1/2$. This may be deduced from [3, Problem 6.13].

Game A is fair only when $p = 1/2$, which corresponds to the case of maximum entropy.

In game B, each choice of p_2 induces a corresponding p_1 for which fairness occurs. These pairs (p_1, p_2) will not correspond to maximum entropy except when $p_2 = 1/2$, which induces $p_1 = 1/2$.

Consider game C with p, p_1, p_2 fixed, so that (1) defines q_1, q_2 as continuous functions of $\gamma \in [0, 1]$ and thus $f(\gamma) := (1-q_1)(1-q_2)^2/(q_1 q_2^2)$ is also a continuous function of γ. Suppose that p, p_1, p_2 are such that games A and B are losing and that there exists a $\gamma = \gamma_0 \in (0, 1)$ such that C is winning. For $\gamma = 0, 1$, we have $(q_1, q_2) = (p_1, p_2)$ and $(q_1, q_2) = (p, p)$ and (2) yields $f(0) > 1$ and $f(1) > 1$. On the other hand, (3) holds for $\gamma = \gamma_0$, so that $f(\gamma_0) < 1$. By continuity there must be at least one $\gamma_1 \in (0, \gamma_0)$ and one $\gamma_2 \in (\gamma_0, 1)$ for which f assumes the value unity.

Fairness entails $f(\gamma) = 1$ and so $q_1 = (1-q_2)^2/[q_2^2 + (1-q_2)^2]$ or

$$1 - q_1 = R(q_2) := q_2^2/[q_2^2 + (1 - q_2)^2]. \tag{9}$$

We may also eliminate γ between (1) for $i = 1$ and $i = 2$ to derive

$$1 - q_1 = L(q_2) := \frac{p - p_1}{p_2 - p_1} q_2 + \frac{p_2(1 - p) - p(1 - p_1)}{p_2 - p}.$$

Thus if the mixed game is fair, it will occur for a value of $q_2 \in (0, 1)$ at which $L(q_2) = R(q_2)$. We readily verify that R is convex with an increasing positive derivative on $(0, 1/2)$ and concave with a decreasing but positive derivative on $(1/2, 1)$.

Suppose as in the example in [2] that $p_2 > p > p_1$ (so that L is strictly increasing), that $p_2(1-p) > p(1-p_1)$ (so that $L(0) > 0$) and that $L(1/2) > 1/2$. Then $L(0) > R(0)$ and $L(1/2) > R(1/2)$ and consequently $L(q_2) > R(q_2)$ throughout $[0, 1/2]$. Further, $L(q_2) = R(q_2)$ can occur for at most two values of $q_2 \in (1/2, 1]$. Thus under the given assumptions there is exactly one $\gamma_1 \in (0, \gamma_0)$ and one $\gamma_2 \in (\gamma_0, 1)$ for which the mixed game is fair. If the associated values of (q_1, q_2) are respectively $(q_1^{(1)}, q_2^{(1)})$ and $(q_1^{(2)}, q_2^{(2)})$, we must have $q_1^{(1)} < q_1^{(2)} < p < q_2^{(2)} < q_2^{(1)}$.

We now demonstrate that the entropy value associated with $q_1^{(2)}, q_2^{(2)}$ must exceed that associated with $q_1^{(1)}, q_2^{(1)}$. Note that the condition $(1 - q_1)(1 - q_2)^2/(q_1 q_2^2) = 1$ for fairness can be interpreted as the unique nontrivial Kolmogorov–cycle condition for the Markov chain C' (see [4, Section 1.5]), so that C' is reversible. This enables us to simplify the expression for S_C in (6).

Reversibility implies that the equilibrium probabilities for C' satisfy the detailed balance equations $\pi_1^C q_2 = \pi_2^C(1 - q_2)$ and $\pi_2^C q_2 = \pi_3^C(1 - q_1)$. Equation (9) and the normalization $\sum_i \pi_i^C = 1$ provide

$$\pi_1^C = (1 - q_2)/S, \quad \pi_2^C = q_2/S, \quad \pi_3^C = (S - 1)/S,$$

where $S := S(q_2) = 1 + q_2^2 + (1 - q_2)^2$. We deduce from (6) that

$$S_C = H_4\left(\frac{(1 - q_2)^2}{S}, \frac{q_2^2}{S}, \frac{q_2}{S}, \frac{1 - q_2}{S}\right) - H_2\left(\frac{S - 1}{S}, \frac{1}{S}\right), \tag{10}$$

where $H_n(p_1, \cdots, p_n) := \sum_{i=1}^n p_i \log(1/p_i)$ for a set $\{p_1, \cdots, p_n\}$ of probabilities summing to unity.

It is readily shown that the second and third arguments of H_4 are strictly increasing functions of q_2 and the first and fourth strictly decreasing. Hence

$$\frac{(1 - q_2^{(1)})^k}{S(q_2^{(1)})} < \frac{(1 - q_2^{(2)})^k}{S(q_2^{(2)})} < 1/4 < \frac{(q_2^{(2)})^k}{S(q_2^{(2)})} < \frac{(q_2^{(1)})^k}{S(q_2^{(1)})} \quad (k = 1, 2).$$

Also for $i = 1, 2$,

$$\frac{(1 - q_2^{(i)})^2}{S(q_2^{(i)})} < \frac{(1 - q_2^{(i)})}{S(q_2^{(i)})} < 1/4 < \frac{(q_2^{(i)})^2}{S(q_2^{(i)})} < \frac{q_2^{(i)}}{S(q_2^{(i)})}.$$

We deduce that if $q_2^{(i)}$ is associated with entropy $H_4^{(i)}$ $(i = 1, 2)$, then $H_4^{(2)} > H_4^{(1)}$. A similar argument shows that $H_2^{(2)} < H_2^{(1)}$. Hence $S_C(q_2^{(2)}) > S_C(q_2^{(1)})$, so the two fair mixed games have different entropies. This leaves open the nature of the connection between fairness and the value of the entropy in the mixed game.

CONCLUSION & OPEN QUESTIONS

We have found expressions for the entropies of the two component losing games in [2] and a winning mix made up from them. For a natural range of parameter values, game A has the greater entropy than games B or C. We have analyzed the structure of the entropy of a mixed game when it is fair. Again for a substantial domain of the parameter space, there are two fair games and we have shown that they have different entropies.

It should be noted that although a fair game corresponds to a reversible Markov chain, such a chain still has an entropy which increases with time unless equilibrium has already been attained (see [4, Section 1.4]).

The expression (10) obtained for the entropy of a fair game is striking and invites further consideration. It also suggests the following.

Problem. To investigate the entropy of an arbitrary reversible Markov chain of form (4), or more generally, its $n \times n$ generalization

$$P = \begin{pmatrix} 0 & r_1 & 0 & \cdots & 0 & 1-r_1 \\ 1-r_2 & 0 & r_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_n & 0 & 0 & \cdots & 1-r_n & 0 \end{pmatrix}.$$

In particular we conjecture that if the numbers $\{r_i, 1 - r_i; 1 \leq i \leq n\}$ are made 'more equal' in an appropriate sense, then the entropy of the chain will be increased. Such a result would constitute an interesting extension of [3, Problem 6.13] and a useful tool for investigating questions such as those considered here.

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