Analytical Explanations of the Parrondo Effect

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Abstract

We give analytical explanations of the so-called Parrondo effect, in which simple coin-flipping games with negative expected value are combined into a winning game. By identifying the previously neglected Parrondo game state parameter, we are able to show that it is possible to obtain a winning game by merely adjusting the state transition probabilities. Our model unravels the often claimed counter-intuitive nature of the Parrondo effect to be no more than tacit interaction between the Parrondo game constituents.

INTRODUCTION

A *Parrondo game* is a combination of two or more simple games in which possibly biased coins are flipped, each with zero or negative expected value, a strategy for alternation between the simple games, and a state. The state determines the probabilities of winning in one or more of the simple games and codes the game history, including the capital of one or more players of the Parrondo game. A winning strategy yields a positive expected value of the Parrondo game, in spite of the constituent simple games having negative expected value: the so-called Parrondo effect. Conditions for the simple games and for the strategies (and implicitly also for the Parrondo game states) were first given by Parrondo *et al.* [1] and the area was recently surveyed in [2]. We give analytical explanations to results previously only approximated by computer simulations, and provide an explicit explanation utilizing the Parrondo game state parameter, of the Parrondo effect.

The original Parrondo game rules [3] combined two games, of which the second was later modified to "present new games where all the rules depend only on the history of the game and not on the capital" [4]. We begin by analyzing this modified game, named B', demonstrating that although the ergodic expected value of B' is negative, it is possible to obtain a winning game by adjusting the state transition probabilities in the game in which B' is mixed with the original Parrondo game named A. We then calculate the optimal strategy for the mixed game. Finally, we analyze the original Parrondo game in an analysis that requires the introduction of a new state parameter: the capital of the player.

GAME B'

The set of possible outcomes of game B' is $\Omega = \{-1, 1\}$, also called *losing* and *winning*. The game history $g_t \in \Omega$ is the outcome of game B' at time t. The probabilities of the outcomes depend on the game history in the following way

$$\begin{aligned} p_{1|-1,-1} &= 9/10 - \epsilon \\ p_{1|1,-1} = p_{1|-1,1} &= 1/4 - \epsilon \\ p_{1|1,1} &= 7/10 - \epsilon \end{aligned} \tag{1}$$

where we use the notation

$$p^t_{ijk\cdots} \doteq Prob[g_t = i, g_{t-1} = j, g_{t-2} = k, \ldots]$$

for the time dependent distribution, $p_{ijk\cdots}$ for the ergodic distribution where $p^t_{ijk\cdots} = p^{t-1}_{ijk\cdots}$, and

$$p_{i|j\cdots} = \frac{p_{ij\cdots}}{p_{j\cdots}}$$

For example, the probability of winning after having lost two simple games is 9/10-ϵ.

GAME B' HAS NEGATIVE EXPECTED VALUE

For the ergodic process, it holds that

$$p_{ij} = \sum_{k\in\Omega} p_{i|jk} p_{jk}, \qquad \sum_{(i,j)\in\Omega^2} p_{ij} = 1 \tag{2}$$

The linear system (1) and (2), has the following solution.

$$\begin{aligned} p_{-1,-1} &= \frac{45 + 210\epsilon + 200\epsilon^2}{198 + 220\epsilon} \\ p_{1,-1} = p_{-1,1} &= \frac{27 + 60\epsilon - 100\epsilon^2}{99 + 110\epsilon} \\ p_{1,1} &= \frac{45 - 230\epsilon + 200\epsilon^2}{198 + 220\epsilon} \end{aligned}$$

The ergodic expected value of the game is:

$$\begin{aligned} \langle g_t \rangle &= \sum_{i\in\Omega} i\, p_i \\ &= \sum_{(i,j,k)\in\Omega^3} i\, p_{i|jk} p_{jk} \\ &= p_{-1,-1}(\frac{2}{5} - 2\epsilon) + (p_{-1,1} + p_{1,-1})(-\frac{1}{2} - 2\epsilon) + p_{1,1}(\frac{4}{5} - 2\epsilon) \qquad (3) \\ &= -\frac{20\epsilon}{9 + 10\epsilon} \qquad (4) \end{aligned}$$

Thus, the game has negative expected value for $\epsilon > 0$.

Mixed with another game, B' can have a higher expected value because the outcome probabilities depend on the Parrondo game history g_t^* rather than the simple game history g_t. Eq. (3) shows that if it holds for the Parrondo state transition probabilities that $p_{1,1}^* \geq p_{1,1}$ and $p_{-1,-1}^* \geq p_{-1,-1}$, then $\langle g_t^* \rangle \geq \langle g_t \rangle$.

The original biased coin-flipping game has outcome 1 with probability $q_1 = 1/2 - \epsilon$, and outcome -1 otherwise (see [3]), and was in [4] mixed with B'. g_t^* is with probability $u = 1/2$ the outcome of the original game, otherwise it is the outcome of B'. The mixed game has positive expected value, i.e. $\langle g_t^* \rangle > 0$ for some $\epsilon > 0$. The fact the simple game B' in this mixed game also has positive expected value goes unnoticed in [4]. More specifically, the negatively biased coin-flipping original game increases the probability $p_{-1,-1}^*$ for two consecutive losses in the mixed game, which in turn increases the expected value of the game B' enough to compensate for the loss suffered from the other simple game. For the mixed game p_{ij}^* it holds that

$$\begin{aligned} p_{ij}^* &= u^2 q_i q_j + u(1-u) p_i^* q_j + u(1-u) q_i p_j^* + (1-u)^2 p_{ij}^* \\ &= u^2 q_i q_j + u(1-u)(\sum_{k\in\Omega} p_{ik}^*) q_j + u(1-u) q_i (\sum_{k\in\Omega} p_{jk}^*) + (1-u)^2 (\sum_{k\in\Omega} p_{i|jk} p_{jk}^*) \quad (5) \\ \sum_{(i,j)\in\Omega^2} p_{ij}^* &= 1 \quad (6) \end{aligned}$$

since p_{ij}^* depends on both of the simple games. The linear system Eq. (1), (5) and (6) has for $u = 1/2$ the solution

$$\begin{aligned} p_{-1,-1}^* &= \frac{4885 + 19530\epsilon + 19200\epsilon^2}{20378 + 5280\epsilon} \\ p_{-1,1}^* &= \frac{-2592 - 1352\epsilon + 10480\epsilon^2}{10189 + 2640\epsilon} \\ p_{1,-1}^* &= \frac{-2622 - 1548\epsilon + 8720\epsilon^2}{10189 + 2640\epsilon} \\ p_{1,1}^* &= \frac{5065 - 20050\epsilon + 19200\epsilon^2}{20378 + 5280\epsilon} \end{aligned}$$

This results in a positive expected value of the Parrondo game

$$\begin{aligned} \langle g_t^* \rangle &= 2p_{1,1}^* - 2p_{-1,-1}^* \\ &= \frac{180 - 39580\epsilon}{10189 + 2640\epsilon} \\ &= \frac{90}{10189} - \frac{201877910}{103815721}\epsilon + O(\epsilon^2) \end{aligned}$$

The positive expected value is simply and intuitively due to changing the weights p_{ij} in the weighted sum in Eq. (3), which shows the tacit dependence between the simple games.

OPTIMAL MIXING STRATEGIES

Harmer and Abbott [2] have experimentally studied a parameter for the probability of playing the simple games in a Parrondo game, in order to maximize the capital of the player. The optimal state-independent strategy u^* is found by maximizing $\langle g_t^* \rangle$ over $u \in [0,1]$, which is the quotient of two fifth-degree polynomials. For $\epsilon = 1/1000$, $u^* \approx 0.2906$. In a similar manner, the optimal state-dependent strategies can be calculated by defining $\langle g_t^* \rangle$ as a function of the conditional probabilities $p^*_{i|jk\cdots}$.

PARRONDO'S ORIGINAL GAME HAS POSITIVE EXPECTED VALUE

In Parrondo's original game (see [3]), the positive game outcome again depends on the tacit game interaction with a state parameter, in this case the accumulated capital. The game outcome at time t is $g_t \in \{-1, 1\}$. The winning probabilities depend on the accumulated capital $C_t = C_{t-1} + g_{t-1}$. The conditional transition probabilities are given by

$$p_{1|;0} = \frac{1}{2}P + \frac{1}{2}P_1 \qquad p_{1|;1} = p_{1|;2} = \frac{1}{2}P + \frac{1}{2}P_2 \tag{7}$$

$$P = 1/2 - \epsilon, \qquad P_1 = 1/10 - \epsilon, \qquad P_2 = 3/4 - \epsilon \tag{8}$$

where we use the notation

$$p^t_{ijk\cdots;l} = Prob[g_t = i, g_{t-1} = j, g_{t-2} = k, \cdots, C_t \equiv l \pmod{M}]$$

and skip the t for the ergodic transition probabilites, and denote conditional probability $p_{i\cdots|j\cdots} = p_{i\cdots}/p_{j\cdots}$. Hence, $p_{1|;0}$ is the probability of winning when the capital $C \equiv 0 \pmod{M}$. For $M = 3$ we observe that

$$p_{;0} = p_{-1;1} + p_{1;2} \qquad p_{;1} = p_{-1;2} + p_{1;0} \qquad p_{;2} = p_{-1;0} + p_{1;1} \tag{9}$$

and since

$$p_{i;j} = p_{i|;j}p_{;j} \tag{10}$$

we can solve for the unknown $p_{;i}$ in the linear system (7), (8), (9), and (10) which for $\epsilon = 1/1000$ has the solution

$$p_{;0} = \frac{95672}{276941} \qquad p_{;1} = \frac{10046}{39563} \qquad p_{;2} = \frac{110947}{276941} \tag{11}$$

The unconditional probability of winning is

$$p_{i;} \;=\; \sum_{0 \le j < M} p_{i|;j} p_{;j} \tag{12}$$

and hence, from (7), (8), (11), and (12), the probability of winning is

$$p_{1;} = \frac{17714723}{34617625} \approx 0.5117$$

and therefore

$$\langle g_t \rangle \;=\; p_{1;} - (1 - p_{1;}) \approx 0.0234$$

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