LEVEL 5 ELECTROMAGNETIC THEORY AND RFID APPLICATIONS
Part 2: Electromagnetic theory - The differential calculus aproach

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Chapter 1

INTRODUCTION

1.1 Course Definition

The course has the formal title: "Electromagnetic Theory and RFID Systems: Part 2: The differential calulus approach,".

1.2 Objectives

1. The principal aims of this course are:

    - To revise students knowledge of engineering notation for time varying quantities, vector calculus, and elementary electromagnetic field theory.
    - To extend students' knowledge of electromagnetic field theory into the ares of electrodynamics, material media, and engineering applications of the theory.
    - To develop in students a synoptic view in which the charge conservation equation, the source and vortex picture of electromagnetic fields, Maxwell's equation's, Gauss' and Stokes' theorems, and the electromagnetic boundary conditions are seen as a mutually supporting whole.
    - To develop in students a capacity for rapid visualisation of electromagnetic field configurations which arise from known sources within specified boundaries.
    - To understand the laws of electrodynamics in a form which will require in the future no further modification or generalisation.
    - To provide (in Chapter 10) a preview of a powerful principle which will be developed in later years.

2. The course also aims to assist students:

    - To become skilled in the interpretation and use of the vector calculus.
    - To employ correct and consistent terminology.
    - To acquire a precise knowledge of units and dimensions.
    - To see suitable symmetries in Maxwell's Equations.

- To recognise both the *amperian current* and the *magnetic charge* models for magnetisation.

As a preview of material which will occur in later courses, students are introduced to the concept of *retarded potentials* for time varying fields.

The lecture notes contain among the Appendices a number of exercises, and summaries of useful formulae and results.

1.3 Viewpoint

The course attempts to provide an effective synthesis of vector algebra, Maxwell's equations, and Faraday's line of force field pictures, to give students a feel for the possible and impossible in electromagnetic fields, and to give students a knowledge of the basic principles which can be applied to any field problem. In particular the course aims to assist students to see as a *highly integrated and mutually supporting whole* the following concepts and results:

- The charge conservation equation.
- Maxwell's equations in complete form.
- Gauss' and Stokes' theorems.
- The translation of Maxwell's equations between their differential and integral forms.
- Electromagnetic field pictures as developed by Faraday.
- The source and vortex picture of electromagnetic fields.
- Polarisation and magnetisation within media.
- Electromagnetic boundary conditions.

1.4 Assumed Knowledge

Students are *expected* to bring to the course a working knowledge of the following concepts and results:

- Electrostatic fields and potentials for charges in free space.
- Magnetostatic fields from currents in free space.
- The names and units of the four field vectors.
- Simple free space relations between $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{E}}$ and between $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$.
- Gauss' law of electrostatics in free space and its expression as a vector integral equation involving the $\boldsymbol{\mathcal{D}}$ vector.

- Ampère's law of magnetostatics and its expression as a vector integral involving the $\boldsymbol{\mathcal{H}}$ vector.
- Gauss' law of magnetostatics and its expression as a vector integral equation involving the $\boldsymbol{\mathcal{B}}$ vector.
- Faraday's law of electromagnetic induction and its expression as a vector integral equation involving the $\boldsymbol{\mathcal{B}}$ vector.

It is expected that entering students may be still have some confusion as to the separate roles played by $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$ in situations other than free space, and that a mistakenly optimistic linear view of material media may prevail. Clarification of these matters is one of the important objectives of the course.

1.5 Text Books

Two text books with a view sympathetic to that presented in this section of the course are:

1. M. N. O. Sadiku,"Elements of Electromagnetics", Saunders Publishing, (1989).
2. W. H. Hayt,"Engineering Electromagnetics", 5th edition McGraw Hill, (1989).

Neither text is a required purchase for the course.

1.6 Notation

In electrical engineering it is common to develop equations for physical phenomena in which the variables may be any one of the following:

- In dc circuits or static fields, constant real values which directly represent the physical variables under discussion.
- In ac circuits, real functions of time with an arbitrary time variation and possibly containing a dc component, which directly represent physical variables under discussion.
- In ac circuits in which all variables with a time dependence are varying sinusoidally, *phasor* variables, possibly with a spatial dependence but with no time dependence, which represent the magnitude and phase of the sinusoidal quantities.

It is also true that in field theory some variables, such as an electric field, are *vectors* with three cartesian components, while some variables, such a voltage or current, have no spatially directional property and are thus *scalars.* In this discussion it is important to distinguish between *complex numbers* and *vectors.* We will not in this course call a single complex number represented in the Argand diagram a vector. The term *vector* is reserved for reference to quantities which have components in the three-dimensional space in which we live.

Of course we can have, when we wish to represent a field variable in the sinusoidal steady state, a quantity which we say is a *complex vector.* This is simply a collection of three *phasors,* each with a real and imaginary part, which represent the three sinusoidally varying components of the field along the three cartesian axes.

The field vectors can in the general (non-sinusoidal) case have, in addition to their time variation, a spatial variation. In setting out equations, we may for emphasis explicitly show the time or space variation, or we may for compactness just write the symbol for the variable with the functional variation understood. Both of these things are done in the equation for an electric field vector

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}(x, y, z, t). \tag{1.1}$$

This sample equation shows that we are using in these notes **bold face** type to represent vector quantities. The use of non bold face type to represent scalar variables is shown by the equation

$$v = v(t) \tag{1.2}$$

or even the equation

$$v = V_m \cos(\omega t) \tag{1.3}$$

in which we are for the moment representing a sinusoidally varying quantity without using phasors.

It is important in study of the subject to be clear as to which type of variable is being used at each stage of any analysis, as to give as an answer to a problem a complex number for a variable which is inherently real, such as a power or the instantaneous value of a current, is to commit a nonsense.

To aid in this understanding, it is highly desirable to be able to use a different type face for scalar and vector quantities, and also a different type face for real time varying quantities and for the phasors which can represent the amplitude and phase of sinusoidally varying quantities, whether vector or scalar.

These notes have been prepared using the LaTeX program which provides a sufficient range of type faces for us to be able to distinguish easily between the notations for complex phasor and real time varying quantities. Both the relation between complex phasors and the real time varying physically meaningful variables, and the notation used for, are illustrated by the following equations

$$v(t) = \Re\left\{\mathrm{V} e^{j\omega t}\right\} \tag{1.4}$$

$$\boldsymbol{\mathcal{E}}(x, y, z, t) = \Re\left\{\mathbf{E}(x, y, z) e^{j\omega t}\right\} \tag{1.5}$$

In the first equation we see that we have been able to distinguish between the real and phasor variables by using lower case for the scalar real time-varying quantity and upper case for the phasor representing it. In the second equation, where a vector is involved, a Calligraphic upper case $\boldsymbol{\mathcal{E}}$ has been used on the left hand side to represent the real time-varying vector, while on the right hand side the complex vector phasor is represented by a Roman upper case $\mathbf{E}$. The different types makes the different quantities quite distinct. The other form of notation to be aware of is the way in which components of both real time varying and complex phasor vectors are shown. For the former we will use subscripted upper case type with no bold face, for example E_t, while for the latter we will use subscripted upper case Roman type with no bold face, for example $\mathrm{E_t}$.

The interpretation of variables can also be aided by recognising on the context in which they are used. Some simple rules which aid in determining between phasors and real time varying quantities are as follows.

- An equation in which time derivatives appear is in terms of the real physical variables.
- An equation involving $j = \sqrt{-1}$ is in complex phasors.
- Both sides of an equation must be in variables of the same type; we do not put time domain variables equal to frequency domain variables.

The equations above show another and vital part of field theory notation. Both equation 1.4 and equation 1.5 show that the phasors used have a magnitude equal to the *peak* rather than *rms* values of the sinusoidal variables which they represent.

The representation by phasors denoting peak rather than rms values is a convention universally observed in books on field theory and in the majority of books on communications. It stands in contrast to the almost universal convention of employing in power systems engineering phasors in which the magnitude is equal to the rms value of the quantity being represented. In the power engineering context, equation 1.4 would not be true.

A consequence of the use in field theory of phasors of which the magnitude represents the peak value is that calculations involving power will require a factor of 1/2, not present in power systems calculations, to be inserted. For the power P dissipated in a resistor R by a phasor current I we write:

$$P = \frac{1}{2}|\mathrm{I}|^2 R \tag{1.6}$$

The requirements of good manners prevent us from saying whether we believe the field theory or the power systems tradition is the more sensible.

1.7 The Four Field Vectors

For engineering purposes, it is necessary to deal with electromagnetic fields in terms of four distinct field vectors, invariably denoted by the symbols $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$.

Although the formal definition of these vectors will not occur until later chapters, we take the opportunity to list below the engineering names and the standard international units for them.

$\boldsymbol{\mathcal{E}}$	Electric Field	$\mathrm{Vm^{-1}}$
$\boldsymbol{\mathcal{H}}$	Magnetic Field	$\mathrm{Am^{-1}}$
$\boldsymbol{\mathcal{D}}$	Electric Flux Density	$\mathrm{Cm^{-2}}$
$\boldsymbol{\mathcal{B}}$	Magnetic Flux Density	$\mathrm{Wbm^{-2}}$

These names and units *should be committed firmly to memory.* Possible different usage of the term *magnetic field* in Physics courses should be noted (but not adopted) so that confusion does not arise.

1.8 Limited Validity Equations

The exposition of electromagnetic theory to follow will begin in the early chapters with a description of electric and magnetic fields which are *constant in time*, and which arise from *sources which are in free space*. We will only proceed to a description of time-varying fields and fields in the presence of material media after the simple situations have been understood.

As a result of this simple beginning, some of the equations to be produced in earlier chapters will be of validity limited to the restricted context of the particular chapter, and will be invalid in the more general contexts of later chapters. On the other hand, some of the equations to be produced in earlier chapters will, more or less as a matter of accident, remain valid in more general contexts.

It seems desirable that students be warned, as new equations appear, as to which of them will need to be modified for more general contexts, and which of the equations will survive the transitions to those more general contexts without evident modification. For this reason an effort will be made to mark, when they appear, the equations which will later need modification with the symbol [**LVE**], standing for limited validity equation. As our search for such instances may not have been perfect, it is not claimed that every occurrence of such equations will have been marked.

Chapter 2

REVIEW OF VECTOR ALGEBRA

2.1 Vector Products

2.1.1 Products of Two Vectors

In vector algebra the scalar product $\mathbf{a} \cdot \mathbf{b}$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ at an angle θ is defined as the scalar $ab\cos\theta$.

The vector product $\mathbf{a} \times \mathbf{b}$ is defined as the vector which has magnitude $ab\sin\theta$ and is in a direction perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$, with the direction sensed by the right hand rule.

In Cartesian co-ordinates the vector product may be evaluated using the formal expansion of the determinant shown in equation 2.1.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{2.1}$$

2.1.2 Products of Three Vectors

The two possible products of three vectors are the scalar triple product and the vector triple product.

The scalar triple product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is invariant under interchange of the dot and cross, and also under cyclic interchange of the three vectors. These properties are illustrated in equation 2.2.

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} \tag{2.2}$$

In the Cartesian co-ordinate system shown above the scalar triple product may be evaluated by the expansion of the determinant as shown in equation 2.3.

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{2.3}$$

The vector triple product has the expansion in terms of its component shown in equation 2.4.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{2.4}$$

2.2 Polar Co-ordinates

2.2.1 Cylindrical Co-ordinates

The standard cylindrical polar co-ordinate system is illustrated in Figure 2.1. The relations between the cartesian and cylindrical co-ordinates are given in equation 2.5.

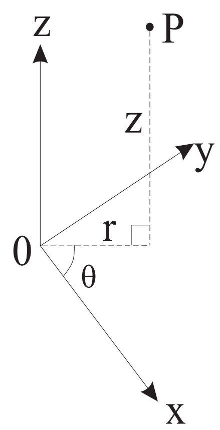


Figure 2.1: Cylindrical polar co-ordinate system

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{2.5}$$

The standard spherical polar co-ordinate system is illustrated in Figure 2.2. The relations between the cartesian and spherical polar co-ordinates are given in equation 2.6.

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2.6}$$

2.3 Vector Differential Operators

2.3.1 Gradient of a Scalar

When $f(x, y, z)$ is a scalar function of position we can form the vector function of position

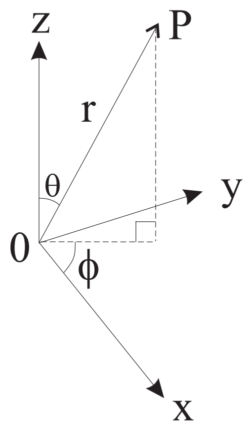


Figure 2.2: Spherical polar co-ordinate system

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{2.7}$$

which is a vector normal to the surface $f = constant$.

2.3.2 Divergence of a Vector

When $\mathbf{F}$ is a vector function of position we can form the scalar function of position

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{2.8}$$

If $\nabla \cdot \mathbf{F} = 0$ everywhere, we say $\mathbf{F}$ is *solenoidal.* The vector field is then easy to picture as the vortex type field illustrated in Section 2.5.

2.3.3 Curl of a Vector

When $\mathbf{F}$ is a vector function of position we can form another vector function of position by formally expanding the determinant

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{2.9}$$

When $\nabla \times \mathbf{F} = 0$ everywhere, we say the vector is *irrotational,* and the vector is easy to picture as the source type field illustrated in Section 2.5. It frequently happens that $\mathbf{F}$ is solenoidal and irrotational nearly everywhere, but it is still easy to picture in terms of what happens in those places where the divergence or curl is different form zero.

2.3.4 Essence of Divergence and Curl

The essential nature of a divergence can to some extent be discovered by examining the formula

$$\text{div}\,\mathbf{F} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}). \tag{2.10}$$

This formula states that for a vector field to have a divergence, at least one component of the vector must increase in magnitude when we progress in the direction of that component. Of course a vector field may have component which increases in magnitude as we progress in that direction and still have no divergence because it has other components which decrease as we progress in their own direction, so as to cancel in the sum above. The essential nature of a divergence is also illustrated by Gauss' theorem in Section 2.4.3 below.

The essential nature of a curl can to some extent be discovered by examining the formula

$$\mathbf{curl}\,\mathbf{F} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \tag{2.11}$$

This formula states that for a vector field to have a curl, at least one component of the vector must increase in magnitude when we progress in a direction orthogonal to the direction of that component. Of course a vector field may have a component which increases in magnitude as we progress in a direction orthogonal to that of the component and still have no curl because it has other components which decrease as we progress in a direction orthogonal to them, so as to cancel in the expressions above. The essential nature of a curl is also illustrated by the concept of *circulation* described in Section 2.4.4 and by Stokes' theorem stated in Section 2.4.5.

2.3.5 Vector Identities

The following five vector identities are easily established by the method of writing out the components of each side in a cartesian co-ordinate system.

$$\text{div}\,\mathbf{grad}\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{2.12}$$

$$\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\mathbf{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathbf{curl}\,\mathbf{B} \tag{2.13}$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{A} = \mathbf{grad}\,\text{div}\mathbf{A} - \nabla^2\mathbf{A} \tag{2.14}$$

$$\text{div}(\psi\mathbf{A}) = \psi\,\text{div}\mathbf{A} + \mathbf{A}\cdot\mathbf{grad}\psi \tag{2.15}$$

$$\mathbf{curl}(\psi\mathbf{A}) = \psi\,\mathbf{curl}\mathbf{A} - \mathbf{A}\times\mathbf{grad}\psi \tag{2.16}$$

The first three are very frequently used in electromagnetic theory, *and should be committed to memory.* The last two have been included for completeness, but will not be used in this course.

2.3.6 Exercise

The formula in equation 2.14 for **curl curl A** will be useful in our derivation of the wave equation for electromagnetic radiation in free space. Prove the relation by writing out each side of the equation in Cartesian components.

2.3.7 Operators in Curvilinear Co-ordinates

Expressions for the vector differential operators div, **grad** and **curl** are given in both sets of polar co-ordinates on the back page of the recommended texts, and indeed in many other text books. They are provided for spherical polar co-ordinates in Appendix B of these notes.

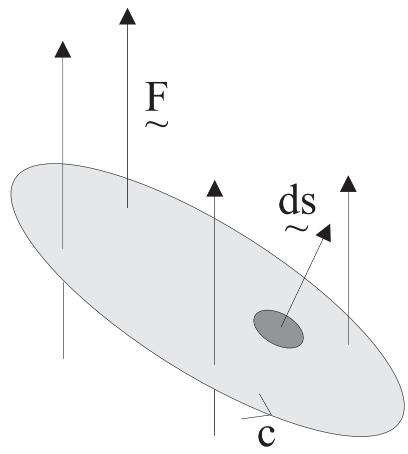


Figure 2.3: Vector field crossing surface S

2.4 Vector Integral Theorems

2.4.1 Potential Theorems

When a scalar potential function ψ exists the integral of its gradient *along any path* between two points given by

$$\int_a^b \nabla\psi \cdot d\mathbf{r} = \psi(b) - \psi(a) \tag{2.17}$$

It follows that for any closed path

$$\oint \nabla\psi \cdot d\mathbf{r} = 0 \tag{2.18}$$

If a vector function $\mathbf{F}$ has the property that $\oint \mathbf{F} \cdot \mathbf{dr} = 0$ for all paths, then there exists a scalar function ψ of position such that $\mathbf{F} = \nabla\psi$, where ψ is uniquely defined except for a constant of integration.

2.4.2 Flux of a Vector

When a vector field intersects a possibly non-planar surface S of which $\mathbf{ds}$ is an element of the surface area as shown in Figure 2.3, we call $\int_S \mathbf{F} \cdot \mathbf{ds}$ the flux Φ of the vector $\mathbf{F}$ flowing through the surface S in the direction defined by the vector $\mathbf{ds}$. In the definition it is essential to define a positive direction for the vector $\mathbf{ds}$.

The positive direction may in general be arbitrarily chosen, (although it must be stated), but in the context of both Gauss' and Stokes' theorems described below, some conventions are required to be observed in that choice. In particular, in Stokes' theorem, it is necessary to define the sense of the surface S in relation to the sense of the boundary contour C shown in the figure. In Gauss' theorem, which applies when the surface is closed and the boundary C shown does not exist, the positive direction for the element of surface area $\mathbf{ds}$ is chosen as outward from the volume enclosed by the surface.

2.4.3 Gauss' Theorem

Consider a closed surface S, of which $\mathbf{ds}$, *sensed outward,* is a vector element of surface area, enclosing a volume v, as shown in Figure 2.4. If $\mathbf{F}$ is an arbitrary vector field, it has been shown by Gauss that

$$\oint_S \mathbf{F} \cdot \mathbf{ds} = \int_v \nabla \cdot \mathbf{F} \, \mathrm{d}v. \tag{2.19}$$

2.4.4 Circulation of a Vector

When a closed path C is defined in a vector field $\mathbf{F}$ as shown Figure 2.5 we may define the *circulation* Γ of $\mathbf{F}$ around the contour C as

$$\Gamma = \oint_C \mathbf{F} \cdot \mathbf{dr} \tag{2.20}$$

It is clear that the circulation depends upon the shape and position of the contour in the field, and upon the direction in which it is traversed in the performance of the line integral. Thus it is necessary to define a positive direction for the contour. This has been done in Figure 2.5 by means of the arrow shown. Again, as with the definition of the positive direction for a surface, the positive direction may be arbitrarily chosen, but in the context of Stokes' theorem discussed below, a restriction applies.

2.4.5 Stokes' Theorem

When a vector field intersects a non-closed surface S bounded by a contour C as shown in Figure 2.3, and the sense of the contour is chosen so that it is related to the sense of the surface area by the right hand rule, it has been shown by Stokes that

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} \tag{2.21}$$

Notice that the line integral is a closed one, but the surface integral is not closed.

Stokes' theorem may be put into words as stating that the circulation of a vector field around a contour bounding a surface is equal to the flux of the curl of that field through that surface.

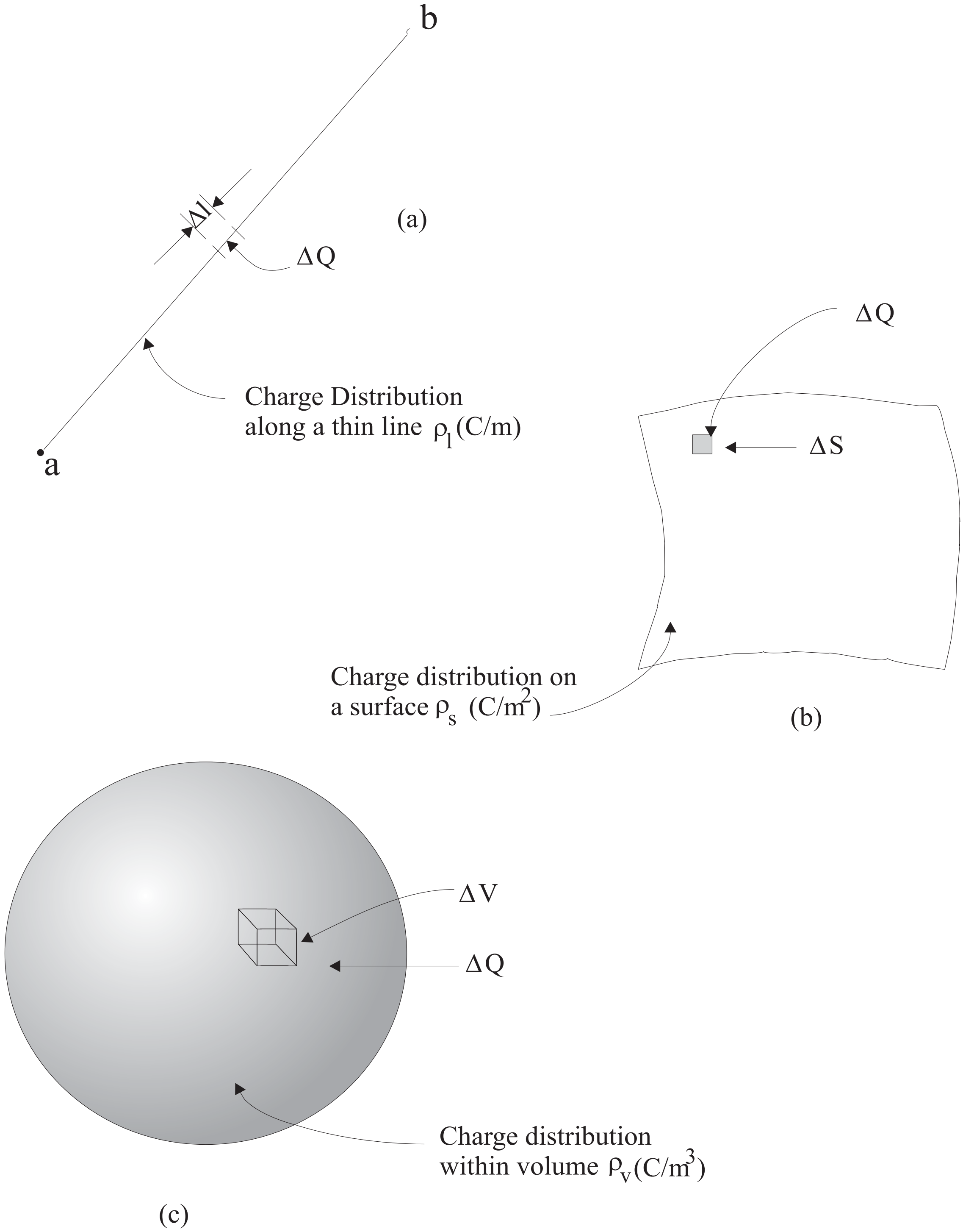


Figure 2.4: A Closed Surface for Gauss' Theorem

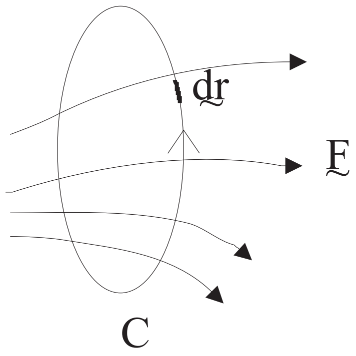


Figure 2.5: Circulation of a vector field around a contour

This leads to the concept of a curl as a vector which gives, for each co-ordinate direction, the circulation per unit area for a small loop placed in a plane perpendicular to the co-ordinate direction.

2.5 Sources and Vortices

The importance of the concept of sources and vortices in a vector field is that rests upon the facts that:

- An ability to form in the mind electromagnetic field pictures appears to be an essential ingredient to mastering electromagnetic theory.
- The source and vortex concept provides a basis for picturing the electromagnetic field created in a wide range of situations.
- The sources and vortices of a field are described in a mathematical sense by the divergence and curl derivatives.
- Maxwell's equations (which are the laws of electrodynamics) are in their differential form direct statements about the divergence and curl of the field vectors.

2.5.1 Source-type Fields

Within a small region, a field is considered to be source-type if there is, as illustrated in Figure 2.6, a net flux of the field which emerges from that region.

In the light of Gauss' theorem, a field will only be source-type within a region if the average value of the divergence of the field vector inside that region is different from zero.

The archetypical example of a source-type field is the electrostatic field which is caused by a distribution of electric charge within the field region.

A field can very well exist within a region, and may have a spatial variation within that region, without having a divergence. An example is the electrostatic field which is created within one region by charges placed within a separate region. In such a case it is not proper to classify the field in the empty region as source type; it is only source type within the charge-containing region.

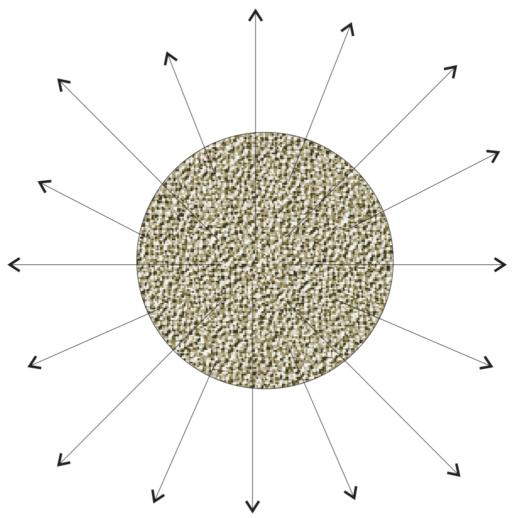


Figure 2.6: Illustration of a source type field

2.5.2 Vortex-type Fields

Within a small region, a field is considered to be vortex-type if there is, as illustrated in Figure 2.7, a net circulation of the field around a contour within that region.

In the light of Stokes' theorem, a field will only be vortex-type within a region if the average value of some component of the curl of the field vector inside that region is different from zero.

The archetypical example of a vortex-type field is the magnetostatic field which is caused by a distribution of electric current within the region.

A field can very well exist within a region, and may have a spatial variation within that region, without having a curl. An example is the magnetostatic field which is created within one region by currents placed within a separate region. In such a case it is not proper to classify the field in the empty region as vortex type; it is only vortex type within the current-carrying region.

2.5.3 Helmholz Theorem

It has been established by Helmholz that a vector field is completely specified by its sources and vortices, ie by its divergence and **curl**. More precisely, if div $\mathbf{F}$ and $\mathbf{curl}\,\mathbf{F}$ are known everywhere within a region, and the field itself is known on the boundary of the region, we can split up as $\mathbf{F}$ into two parts $\mathbf{F}_1$ and $\mathbf{F}_2$ where $\mathbf{F}_1$ and $\mathbf{F}_2$ have properties defined below.

Firstly, $\mathbf{F}_1$ is a *solenoidal* field resulting from the vortices of $\mathbf{F}$, ie $\operatorname{div}\mathbf{F}_1 = 0$ and

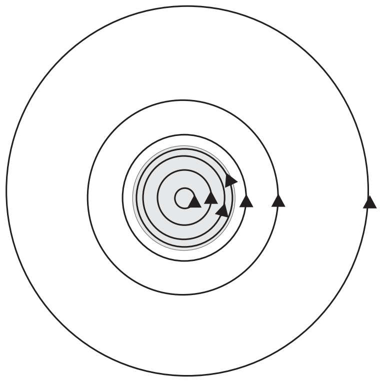


Figure 2.7: Illustration of a vortex type field

$\mathbf{curl}\,\mathbf{F}_1 = \mathbf{curl}\,\mathbf{F}$. It can be further shown that this part of the field can be derived from a vector potential $\mathbf{A}$ such that $\mathbf{F}_1 = \mathbf{curl}\,\mathbf{A}$.

Secondly, $\mathbf{F}_2$ is an *irrotational* field resulting from the sources of $\mathbf{F}$, ie $\mathbf{curl}\,\mathbf{F}_2 = 0$ and $\mathrm{div}\,\mathbf{F}_2 = \mathrm{div}\,\mathbf{F}$. It can be further shown that this part of the field can be derived from a scalar potential ψ such that $\mathbf{F}_2 = \mathbf{grad}\,\psi$.

As stated above, the Helmholz theorem requires a knowledge of the sources and vortices of the field within a region, and a knowledge of the field itself on the boundary of that region. If the region is the whole of space, a knowledge of the behaviour of the field as the field point tends to infinity is required. We will in later parts of the theory find it convenient to introduce a scalar potential and a vector potential for describing the electromagnetic field.

Chapter 3

ELECTROSTATICS

3.1 Charge and Current Descriptors

The primary causes of electric and magnetic fields, at least in free space, may be said to be charges and currents. It is therefore desirable to begin an understanding of electromagnetic theory with a clear picture of how charge or current may be distributed in space, particularly in various idealisations in which charge or current may be restricted to lie in a surface or on a line.

It is useful at this point to introduce the commonly used notation for the descriptors which define charges and currents in a three dimensional volume, on a surface area, and on a line, and to give an indication of the units or dimensions in each case. These are set out in Tables 3.1 and 3.2 on the following two pages, and are illustrated in Figures 3.1 and 3.2 which accompany those Tables.

Students are advised to understand very clearly these concepts, and to pay particular attention to the units of the various descriptors, which in some cases appear to be in conflict with the names.

3.2 Charge Conservation Equation

One of the most important parts of our theoretical framework, and of the mental pictures by means of which it is understood, is the conservation of charge concept. This concept has two aspects, firstly that current is obtained by moving charge around, and secondly that in the process, charge is neither created nor destroyed. As a result, if there is a nett outflow of current from a region, there will be a consequential reduction of charge within that region.

We begin with a further illustration of the concept of volume current density in Figure 3.3, in which a volume current density $\mathcal{J}$ has a component J_x in the x direction, so that the current density produces through the left hand face of area $\Delta y \Delta z$ of the rectangular box of sides Δx, Δy, and Δz a current of magnitude $J_x \Delta y \Delta z$ which enters the box.

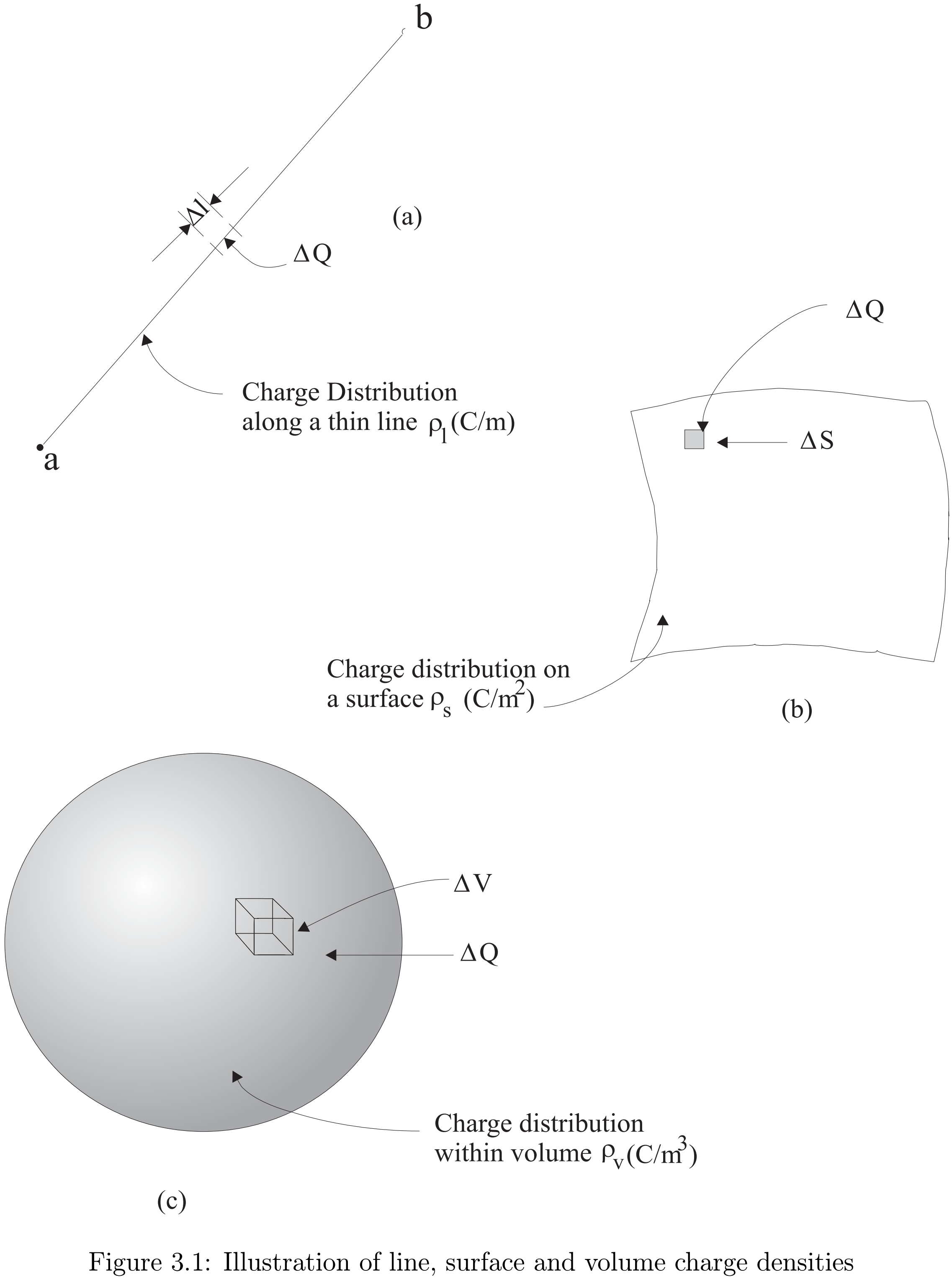


Figure 3.1: Illustration of line, surface and volume charge densities

DESCRIPTOR	SYMBOL	UNITS
Line charge density	ρ_l	Cm^{-1}
Surface charge density	ρ_s	Cm^{-2}
Volume charge density	ρ or ρ_v	Cm^{-3}

Table 3.1: Charge density descriptors

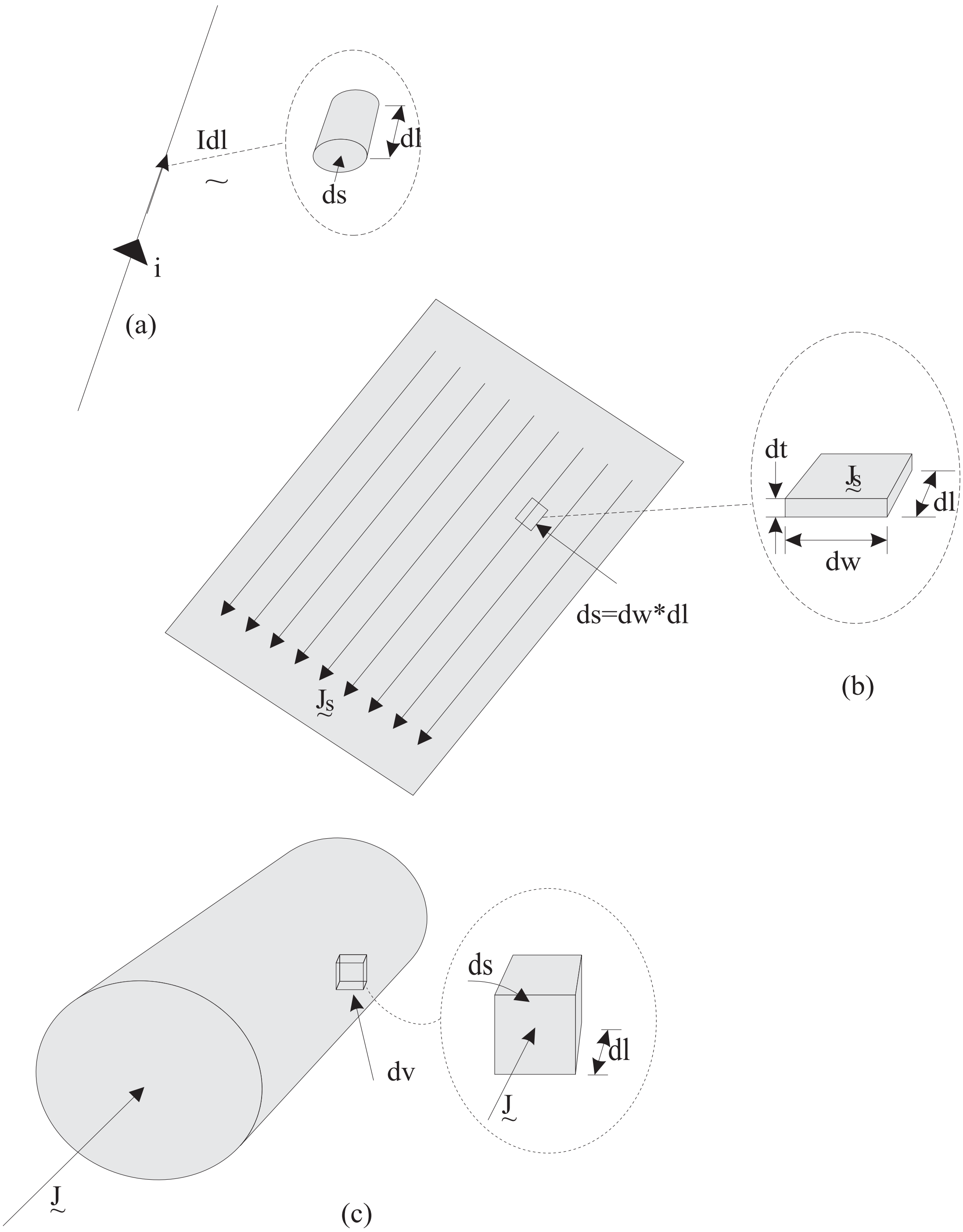


Figure 3.2: Illustration of filamenatary, surface and volume currents

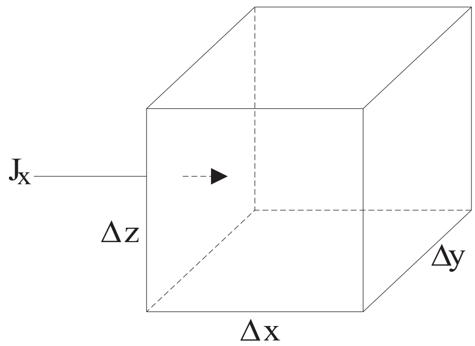


Figure 3.3: Current flow into a box

If the charge density is non-uniform, as illustrated in Figure 3.4, there will be an amount of $J_x \Delta y \Delta z$ charge per unit time entering the box at the left face, and an amount of $(J_x + \frac{\partial J_x}{\partial x}\Delta x)\Delta y \Delta z$ charge per unit time leaving the box at the right face. The nett outflow of charge per unit time via those two faces is thus $\frac{\partial J_x}{\partial x}\Delta x \Delta y \Delta z$. Considering all six faces, we obtain a nett outflow of charge per unit time of

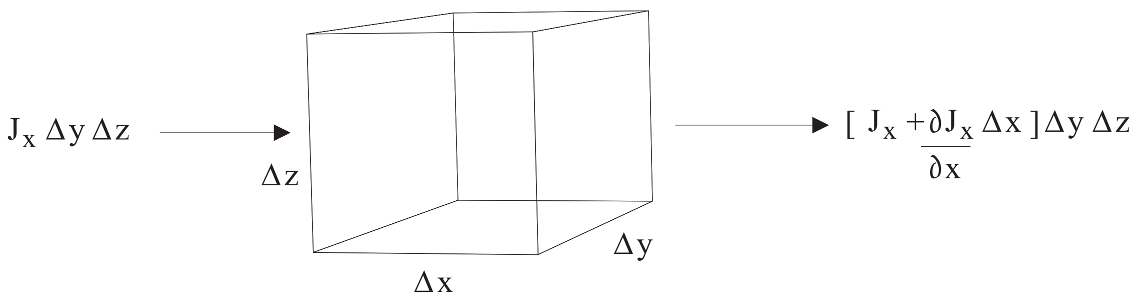


Figure 3.4: Current flow out of the box

$$\frac{\partial J_x}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial J_y}{\partial y}\Delta y \Delta z \Delta x + \frac{\partial J_z}{\partial z}\Delta z \Delta x \Delta y. \tag{3.1}$$

Since charge is neither created nor destroyed, this should be equal to the rate of decrease of charge still within the volume, ie

$$-\frac{\partial \rho_v}{\partial t}\Delta x \Delta y \Delta z \tag{3.2}$$

Combining these results and dividing by a common factor equal to the volume of the box gives the charge conservation equation below.

$$\nabla \cdot \boldsymbol{\mathcal{J}} + \frac{\partial \rho_v}{\partial t} = 0 \tag{3.3}$$

Frequently the subscript is omitted from the charge density and the result is written

$$\nabla \cdot \boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{3.4}$$

The result as derived above is in the time domain. In the frequency domain, the result is written

$$\nabla \cdot \mathbf{J} + j\omega\rho = 0 \tag{3.5}$$

Although this result looks to be very similar to the one preceding it, there is a distinct difference in the meaning of most of the symbols which we have not been able to signify by any change in the type face. In equation 3.4 the symbols $\boldsymbol{\mathcal{J}}$ and ρ represent real, time-varying, current and charge densities. In equation 3.5 the symbols $\mathbf{J}$ and ρ represent complex, non-time varying phasors.

The time domain and frequency domain forms of the charge conservation equation above can be transformed by Gauss' theorem to the integral form equations 3.6 and 3.7 below.

$$\oint_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} + \int_v \frac{\partial \rho}{\partial t} dv = 0 \tag{3.6}$$

$$\oint_S \mathbf{J} \cdot \mathbf{ds} + j\omega \int_v \rho dv = 0 \tag{3.7}$$

3.3 Elementary Electrostatics

3.3.1 Coulomb's Law in Free Space

The force exerted by charge q_1 on charge q_2 distant r from it in free space as shown in Figure 3.5 is

Figure 3.5: Point charges in free space

$$\boldsymbol{\mathcal{F}} = \frac{q_1 q_2 \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \tag{3.8}$$

In the above equation, ϵ_0 is the dielectric permittivity of free space and has the value 8.854 pFm^{-1}.

3.3.2 Electrostatic Field

By defining the electrostatic field at a point the force per unit test charge placed at that point we find that an isolated charge q placed at the origin will produce an electrostatic field $\boldsymbol{\mathcal{E}}$ at a point $\mathbf{r}$ equal to

$$\boldsymbol{\mathcal{E}} = \frac{q\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \tag{3.9}$$

3.3.3 Faraday's Representation

We note from equation 3.8 that the electrostatic force is, as are several other forces in nature, an inverse square central force. It may be shown that such forces may be represented by *field lines* or *lines of force* which fill the space where the force is found and have the properties:

- Their direction at any point gives the direction of the force at that point.
- The number of field lines per unit area crossing at any point a plane drawn perpendicular to the direction of the field at that point is proportional to the magnitude of the field at that point.

The constant of proportionality relating the number of lines of force per unit area and the strength of the field is an arbitrary choice of the person defining the representation. It was realised by Faraday that defining lines of force in that way leads, for an inverse square central force such as the electrostatic force, to a situation where field lines originate or terminate only on charges, and that in the regions between charges, the continuously drawn (or, more commonly, imagined) lines will bend converge or diverge naturally so that their density per unit area intersecting any locally perpendicular plane will correctly represent the magnitude of the field. By this realisation Faraday made a great contribution to our ability to *visualise* an electrostatic field, by forming mental pictures in three dimensions according to his rules.

The difficulties of representing three dimensional pictures, especially of field lines, on paper has led to a situation where powerful though his concept is, Faraday's definition of lines of force is not always employed in text books illustrating electric (or magnetic) fields. In particular fields are sometimes illustrated in a convention in which the strength of the field is proportional to the length of vector drawn to represent both the magnitude and direction of the field. Although this is a legitimate representation in its own right, it is not the Faraday representation, and does not have the property that the field lines will only originate or terminate any charges.

This discussion so far has been of techniques for representing the electrostatic field where field lines always originate or terminate on charges, and never form closed loops. It has been argued that the Faraday representation is the superior representation in terms of which such fields can be visualized in three dimensions. The question arises of whether the Faraday representation is equally suitable for the representation of those electric fields, which can form closed loops, which are induced by changing magnetic flux density.

The essence of the matter is that the representation of any field by Faraday lines or tubes of force is always possible and such lines will need to stop and start only where the field has a non-zero divergence. To represent the electrodynamic field we need merely to relax the electrostatic field conditions that field lines do not form closed loops and allow them to do so, and we have a field representation of full generality. That representation retains the property that field lines will only originate and terminate any charges, and the property that field lines will in regions devoid of changes converge or diverge to correctly represent through their area density and direction the magnitude and direction of the field in a very natural way.

3.3.4 Electrostatic Potential

A further property of an inverse square central force field, such as the electrostatic field, is that the work done in moving a unit charge along any closed path is zero. We know therefore that the electrostatic field is derivable from a potential. Our definition of the potential differs from that given in purely mathematical potential theory in section 2.4 in such a way that a negative sign appears.

Rather than defining the potential at a point as the integral of the work done by the field is moving an object to a point, as might be the convention in a purely mathematical treatment, we define the potential V at a point $\mathbf{r}$ in an electrostatic field as the work *we* do in moving a unit charge from infinity to that point. As the force *we* must exert on the unit charge is $-\boldsymbol{\mathcal{E}}$ (the opposite of the force exerted by the field) we have

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \boldsymbol{\mathcal{E}}(\mathbf{r}) \cdot \mathbf{dr} \tag{3.10}$$

Hence

$$\boldsymbol{\mathcal{E}} = -\mathbf{grad}V \tag{3.11}$$

3.3.5 Electric Flux Density

In the limited context of empty space we may define for an electric field $\boldsymbol{\mathcal{E}}$ an electric flux density vector $\boldsymbol{\mathcal{D}}$ given by

$$\boldsymbol{\mathcal{D}} = \epsilon_0\, \boldsymbol{\mathcal{E}} \quad [\mathbf{LVE}] \tag{3.12}$$

Examination of that equation may reveal that the SI units of $\boldsymbol{\mathcal{D}}$ are Cm^{-2}, although the result may be better appreciated through Gauss' law presented in the next section.

In the limited context of empty space the $\boldsymbol{\mathcal{D}}$ vector may appear as no more than a rescaled version of $\boldsymbol{\mathcal{E}}$. To emphasize the limited validity of the above equation and to dispel the notion of simple proportionality between $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{E}}$, we introduce, in anticipation of a more thorough discussion in Chapter 6, the general definition

$$\boldsymbol{\mathcal{D}} = \epsilon_0\, \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}} \tag{3.13}$$

which applies in the presence of material media, and in which $\boldsymbol{\mathcal{P}}$ is known as the *polarization* of the medium at the point at which $\boldsymbol{\mathcal{D}}$ is defined. In this fully general definition it must be understood that no functional relation between $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{E}}$ necessarily exists, so $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{E}}$ are entirely different quantities, with no proportionality in any way implied. However in free space there is no polarisable medium, so the former equation, 3.12, does apply.

As at this point of the exposition we are still considering electrostatics in free space, some reason for introducing a definition of $\boldsymbol{\mathcal{D}}$ at all should probably be offered. The reason is that Gauss' law for electrostatics, which will be derived in the next section, is best presented in terms of the $\boldsymbol{\mathcal{D}}$ vector, as it will then appear in the form which retains its appearance even when it is generalised in Chapter 6 to the case when material media are present.

3.4 Gauss' Law in Free Space

Gauss' law may be regarded as an alternative statement of the inverse-square central-force property of electrostatic fields, expressed previously by Coulomb's law, and which is so much a part of the Faraday line of force concept discussed above. We will establish the law firstly in integral form, and then proceed via Gauss' theorem of the vector calculus to obtain its equivalent expression in differential form.

3.4.1 Integral Form

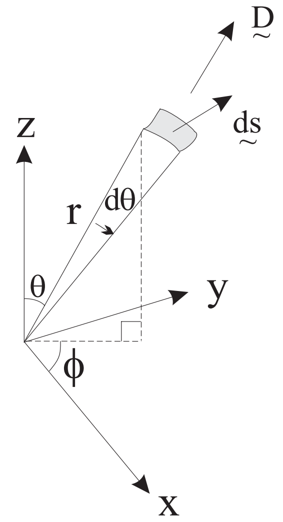


Figure 3.6: Flux from a point charge

We consider first a charge q situated at the origin from which the emerging flux density $\boldsymbol{\mathcal{D}}$ at a position $\mathbf{r}$ has a solely radial component given by

$$D_r = \frac{q}{4\pi r^2} \tag{3.14}$$

We consider next the amount of flux which passes as shown in Figure 3.6 through an infinitesimal surface $\mathbf{ds}$, at the point $P(r, \theta, \phi)$, the element of surface being contained within the limits dr, $d\theta$ and $d\phi$, but the normal to the surface not necessarily being in the radial direction. Forming the scalar product $\boldsymbol{\mathcal{D}}\cdot\mathbf{ds}$ to obtain the flux passing through the surface we obtain

$$\boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = \frac{q\, r d\theta\, r \sin\theta\, d\phi}{4\pi r^2} \tag{3.15}$$

which we notice has now become independent of both the orientation of the vector ds and the distance r of the surface element from the charge. This independence is made the more evident by writing the result first obtained in the form

$$\boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = \frac{q\, d\Omega}{4\pi} \tag{3.16}$$

where $d\Omega$ is the solid angle subtended by the surface element at the position as the charge. In this last form we can see that by adding all contributions to the flux through any closed surface which surrounds the charge, and noting that the total solid angle for all parts of a closed surface is 4π, we have

$$\oint_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = q \tag{3.17}$$

Noting that as the origin has no special relationship to the arbitrarily shaped surface other than to be inside it, we will generalize the above result as applying to a charge q at any point within the closed surface. Finally, when a group of charges, or a volume charge density, forming a total charge Q is enclosed by the surface, we will generalize the result to

$$\oint_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = Q \tag{3.18}$$

$$Q = \int_v \rho \, dv \tag{3.19}$$

3.4.2 Differential Form

Making use of Gauss' theorem

$$\oint_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = \int_v \operatorname{div} \boldsymbol{\mathcal{D}} \, dv \tag{3.20}$$

and the expression for the total charge Q contained within a volume v when a charge density ρ is present, we obtain an equality between the two volume integrals

$$\int_v \operatorname{div} \boldsymbol{\mathcal{D}} \, dv = \int_v \rho \, dv \tag{3.21}$$

We may now consider the volume v to be the infinitesimal volume δv for which the integrals reduce to

$$\delta v \operatorname{div} \boldsymbol{\mathcal{D}} = \delta v \, \rho \tag{3.22}$$

cancelling the common factor δv we obtain

$$\operatorname{div} \boldsymbol{\mathcal{D}} = \rho \tag{3.23}$$

which is the differential form of Gauss' law of electrostatics.

3.5 Common Field Distributions

Although both of the results to be derived in this section may easily be obtained from Gauss' law and some plausible assumptions as to the shapes of the fields resulting, it is the intention that they be derived by the student from Coulomb's law. Such derivation is intended to provide useful practice in handling the mathematics involved.

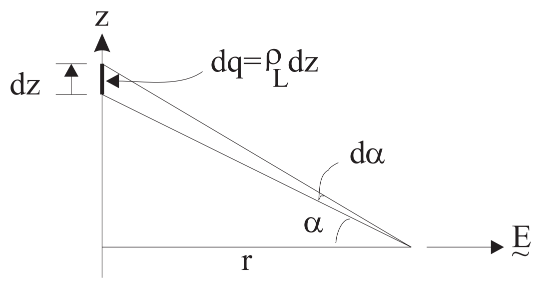


Figure 3.7: Line distribution of charge

3.5.1 Field of a Line Charge

Show as an exercise that for the line charge density ρ_L extending along the z axis from $-\infty$ to ∞ as shown in Figure 3.7, the electric field at a distance r is given by

$$\boldsymbol{\mathcal{E}} = \frac{\rho_L}{2\pi\epsilon_0 r}\hat{\mathbf{r}} \tag{3.24}$$

Derive the result by integration over a series of elemental charges $dQ = \rho_L dz'$, not by Gauss' theorem. It will be found useful to express the contribution from the elemental change in terms of the angle α shown in the figure before integrating over α.

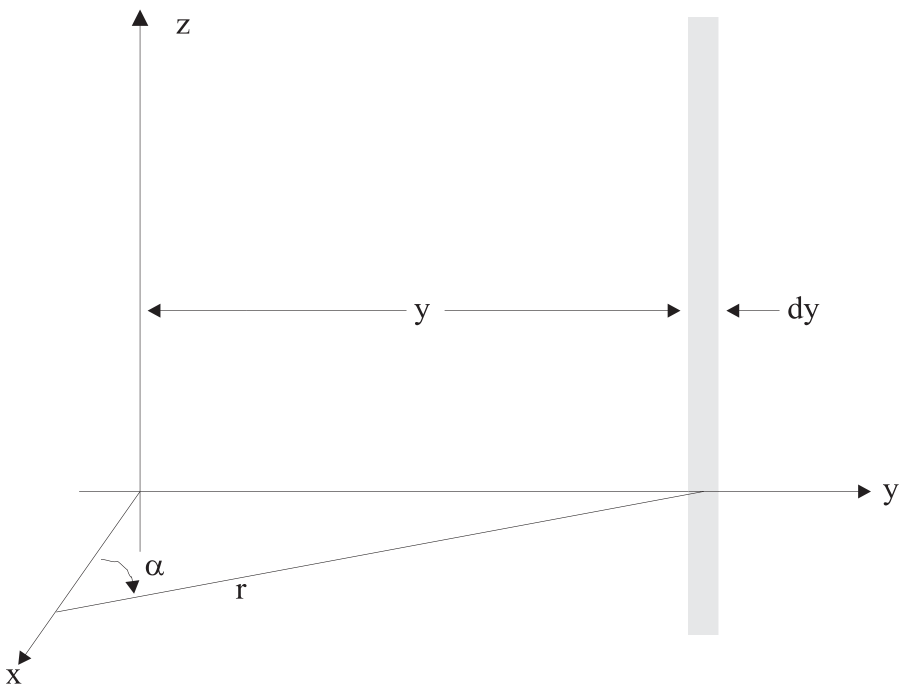


Figure 3.8: Field from a surface charge

3.5.2 Field of a Surface Charge

Figure 3.8 shows a surface distribution of charge ρ_s extending over the entire plane $x = 0$. Show as an exercise that the electric field produced is given by

$$E_x = \frac{\rho_s}{2\epsilon_0} \tag{3.25}$$

Do not use Gauss' law, but derive the result directly, using the result obtained in the section above for the field caused by the line charge contained in the vertical strip of width dy located at y in the $x = 0$ plane, and expressing your result in terms of the angle α before integrating.

$$\boldsymbol{\mathcal{E}} = \frac{\rho_s}{2\epsilon_0}\hat{\mathbf{a}}_x \tag{3.26}$$

3.6 Properties of Dipoles

3.6.1 Definition

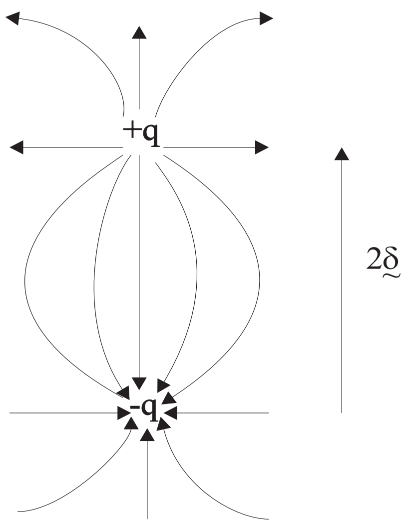


Figure 3.9: An electric dipole

An electric dipole consists of two charges q and $-q$ separated by a distance $2\vec{\delta}$ as shown in Figure 3.9. The *strength* of the dipole is considered to be the vector directed from the negative charge to the positive charge.

$$\mathbf{p} = 2q\vec{\delta} \tag{3.27}$$

The dipole is considered to be *located* at a point mid-way between the charges. An *infinitesimal* dipole is considered to be the result of allowing $\delta \to 0$ while the strength p as defined above remains constant. The field of an infinitesimal dipole is shown in Figure 3.10.

DESCRIPTOR	SYMBOL	UNITS
Filamentary current	I	A
Surface current density	$\mathcal{K}$ or $\mathcal{J}_S$	Am^{-1}
Volume current density	$\mathcal{J}$	Am^{-2}

Table 3.2: Current density descriptors

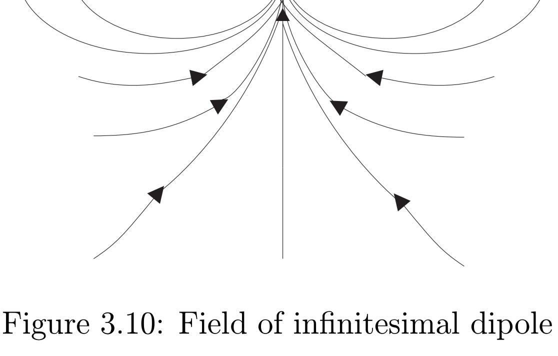


Figure 3.10: Field of infinitesimal dipole

3.6.2 Field Distribution

For the calculation of the field due to an infinitesimal dipole, Figure 3.11 and the spherical polar coordinate system is useful. The procedure is to calculate the fields E^+ and E^- as shown in the figure, and then to resolve them along the directions $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ in spherical polar coordinates. Making the approximation that the angle α subtended at P by the half-dipole is small, ie $\delta << r$, we may show that the dipole field is given by

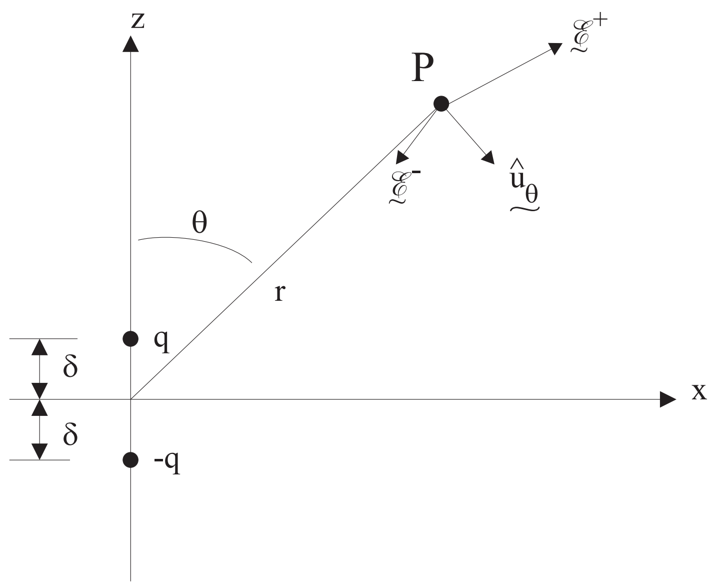


Figure 3.11: Derivation of dipole field

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \quad (3.28)$$

$$E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \quad (3.29)$$

It would be useful to produce these results as an exercise.

3.6.3 Dipole: Source or Vortex?

As the source and vortex viewpoint has been strongly promoted as an aid to the visualisation of electromagnetic fields, it is natural to ask whether the dipole showed be regarded as a source or a vortex. While the dipole remains of finite size as shown in Figure 3.9, the answer is clear: there is a source at the positive charge and a sink at a negative charge. When the dipole becomes infinitesimal, however, as shown in Figure 3.10, the region occupied by the charges becomes too small to examine, but we may be still interested in classifying the external field.

It is easy to show that both the divergence and **curl** of the field at all points other than the point occupied by the dipole are both zero, so in those regions the field is neither source-type nor vortex-type.

We might still be interested in classifying that external field as having been produced by either a source or a vortex located at the dipole, so we might attempt to evaluate both the divergence and **curl** at that point, only to find that our effort is frustrated by the fact that neither of these derivatives exist there. We are led to the conclusion that a dipole is neither a source nor a vortex — it is a singularity of the field caused perhaps by our over-idealisation of the model.

3.6.4 Oscillating Magnetic Flux Loop

In case the temptation to find a way of classifying a dipole as a source or vortex is still active after the above discussion, we might anticipate a result from later chapters by revealing that the field distribution derived in Section 3.6.2 could also be produced by an infinitesimal toroidal ring of time-varying magnetic flux through the process of electromagnetic induction (Faradays' law) as shown in Figure 3.12.

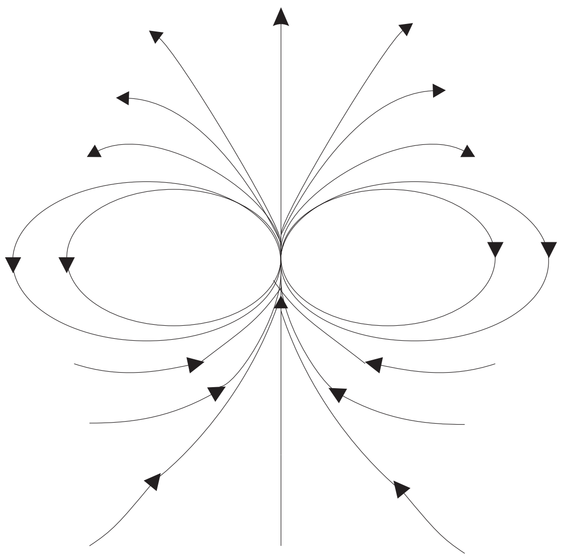


Figure 3.12: Field of infinitesimal oscillating toroidal flux

With this as source, we might be tempted to regard the field as having been caused by a vortex, whereas with the two separated charges as sources, we might be tempted to regard the field as having been caused by sources.

The truth is, however, that neither view is necessary. What we have a glimpse of here is a special case of a *Field Equivalence Principle* which states that an electromagnetic field distribution given over a finite region of space can be caused by more than one (in fact any number of) distribution of charges and currents in the region external to the given region.

This general principle points to the fact that a dipolar field can be regarded as the limiting case of either a pair of source-type fields or a field deriving from a vortex of time varying magnetic flux. This conclusion has bearing on the question of taken up in Chapter 6 of whether we will model the dipolar field of a magnetised atom as arising from circulating currents or from separated magnetic charges. From the perspective just discovered, we appear to have a free choice in the matter.

3.6.5 Torque on a Dipole

It is a simple matter and should be taken as an exercise to show that when a dipole of strength $\mathbf{p}$ is placed in a uniform electric field $\boldsymbol{\mathcal{E}}$ it experiences a torque

$$\boldsymbol{\mathcal{T}} = \mathbf{p} \times \boldsymbol{\mathcal{E}} \tag{3.30}$$

This result obviously generalises to the case of an infinitesimal dipole placed in a not necessarily uniform field.

3.6.6 Force on a Dipole

The result in the section above for the torque on a dipole is well known. Less well known is perhaps the fact that in a non-uniform field, a dipole will also experience a force.

Considering for the moment only the x components of the force, and making use of Figure 3.13 we may see that the x components of the forces at the two ends of the dipole are given by

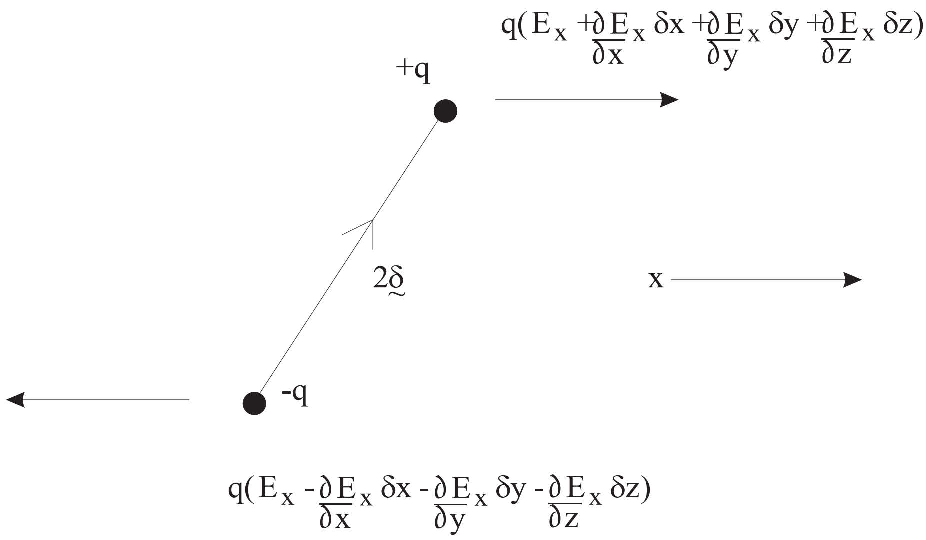


Figure 3.13: Forces on the charges of a dipole in a non-uniform field

$$-q(E_x - \frac{\partial E_x}{\partial x}\delta_x - \frac{\partial E_x}{\partial y}\delta_y - \frac{\partial E_x}{\partial z}\delta_z) \tag{3.31}$$

and

$$+q(E_x + \frac{\partial E_x}{\partial x}\delta_x + \frac{\partial E_x}{\partial y}\delta_y + \frac{\partial E_x}{\partial z}\delta_z) \tag{3.32}$$

Combining these two forces and cancelling appropriate terms gives the x component of the force on the dipole. Similar calculation of the y and z components of the force gives the result

$$\begin{aligned} F_x &= p_x\frac{\partial E_x}{\partial x} + p_y\frac{\partial E_x}{\partial y} + p_z\frac{\partial E_x}{\partial z} \\ F_y &= p_x\frac{\partial E_y}{\partial x} + p_y\frac{\partial E_y}{\partial y} + p_x\frac{\partial E_y}{\partial z} \\ F_z &= p_x\frac{\partial E_z}{\partial x} + p_y\frac{\partial E_z}{\partial y} + p_z\frac{\partial E_z}{\partial z} \end{aligned} \tag{3.33}$$

A shortened way of writing the above equations is in the form

$$\boldsymbol{\mathcal{F}} = (\mathbf{p} \cdot \mathbf{grad})\, \boldsymbol{\mathcal{E}} \tag{3.34}$$

where

$$(\mathbf{p} \cdot \mathbf{grad}) = (p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}) \tag{3.35}$$

is an operator we apply separately to the components E_x, E_y and E_z of $\boldsymbol{\mathcal{E}}$ to obtain the force components F_x, F_y and F_z.

Chapter 4

MAGNETOSTATICS

4.1 Introduction

Knowledge of the magnetic effects of currents, which were first reported by Oersted, was preceded by a considerable time of knowledge of the magnetic effects of magnetised media, and of the magnetic field of the earth itself.

For a full description of magnetic phenomenons it is necessary to make use of three magnetic vectors: *viz.* $\boldsymbol{\mathcal{H}}$ which is known as the *magnetic field*, $\boldsymbol{\mathcal{B}}$ which is known as the *magnetic flux density*, and $\boldsymbol{\mathcal{M}}$ which is known as the *magnetisation* of a material body. The three vectors are related by the equation

$$\boldsymbol{\mathcal{B}} = \mu_0 \left(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}} \right) \tag{4.1}$$

A more complete discussion of this matter will be given in Chapter 6, but we should note here, as we did in relation to dielectric polarisation, that no functional relationship between $\boldsymbol{\mathcal{M}}$ and either of $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$ necessarily exists, and a proportionality between any of two or them is in no way implied.

Elementary treatments of electromagnetism quite often focus on the flux density vector $\boldsymbol{\mathcal{B}}$ to the almost total exclusion of $\boldsymbol{\mathcal{H}}$, with the result that students are left with a significant gap in their knowledge of this most important vector, and with an incorrect view of Ampere's law of which it forms an indispensable part. In an attempt to overcome this problem we will begin our exposition of magnetic effects with a definition and discussion of the magnetic field vector $\boldsymbol{\mathcal{H}}$.

4.2 Magnetic Field Vector

In the full equations of electrodynamics, $\boldsymbol{\mathcal{H}}$ will be seen to be a vector with both sources and vortices. The vortices of $\boldsymbol{\mathcal{H}}$ will be seen to be electric currents, while the sources will be poles formed at the ends of magnetised media, or volume magnetic pole density formed within non-uniformly magnetised media. In free space, however, no magnetic media are present, and $\boldsymbol{\mathcal{H}}$ is a purely vortex-type field caused by electric current.

The definition of $\boldsymbol{\mathcal{H}}$ to be presented below can be regarded in two ways. On one hand it will appear to be a purely mathematical function of the current density. On the other hand it can be regarded as the first step in a two-stage statement of an empirical law, ie the law of force between current-carrying conductors *in vacuo*. It is appropriate to regard it as both of these things, but it must be noted that what we are about to define is only that part of $\boldsymbol{\mathcal{H}}$ (the vortex part) which is caused by currents, and that in the presence of a magnetised medium there will be an additional part (the source part) attributable to the magnetic poles at the ends of the medium.

4.2.1 The Biot-Savart Law

Our definition of $\boldsymbol{\mathcal{H}}$ will be taken from what is known as the law of Biot and Savart, which states that the contribution $\mathbf{d}\boldsymbol{\mathcal{H}}_2$ to the magnetic field $\boldsymbol{\mathcal{H}}_2$ at a point P_2 at position vector $\mathbf{r}_2$, caused by a current element $I_1\mathbf{dr}_1$ at point P_1 at position vector $\mathbf{r}_1$, as shown in Figure 4.1, is given by

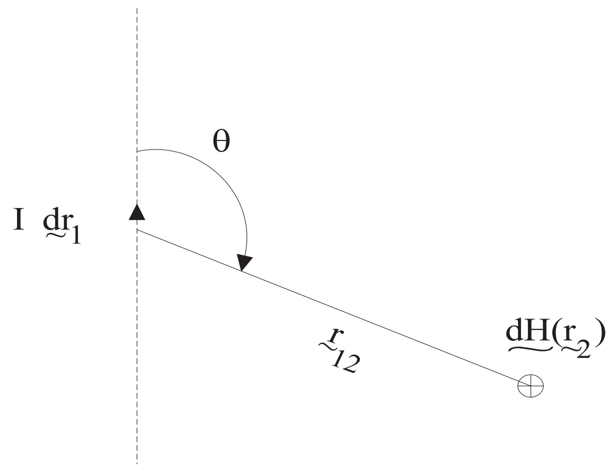


Figure 4.1: Field of a current element

$$\mathbf{d}\boldsymbol{\mathcal{H}}(\mathbf{r}_2) = \frac{I_1 d\mathbf{r}_1 \times \hat{\mathbf{r}}_{12}}{4\pi r_{12}^2} \; [\mathbf{LVE}] \tag{4.2}$$

In the above equation, $\mathbf{r}_{12}$ is the vector $\mathbf{r}_2 - \mathbf{r}_1$ from $\mathbf{r}_1$ to $\mathbf{r}_2$, $\hat{\mathbf{r}}_{12}$ is a unit vector (unit magnitude, no dimensions) in that direction, and r_{12} is the magnitude of $\mathbf{r}_{12}$.

In the figure, the plane of the paper is the plane containing $\mathbf{dr}_1$ and $\mathbf{r}_{12}$. The vector $\mathbf{d}\boldsymbol{\mathcal{H}}(\mathbf{r}_2)$ is perpendicular to that plane, varies in magnitude as $\sin\theta$, and inversely as the square of the distance r_{12}.

4.2.2 Name and Units

We use the term *magnetic field* only for $\boldsymbol{\mathcal{H}}$. The vector $\boldsymbol{\mathcal{B}}$, when we meet it, (perhaps we have already met it), will be called the *magnetic flux density.* The units of $\boldsymbol{\mathcal{H}}$ are obviously, from the definition, Am^{-1}.

4.2.3 Field of a Complete Circuit

For a closed circuit following a contour C on which $\mathbf{r}_1$ is a position vector, the total magnetic field at a point $\mathbf{r}_2$ is

$$\boldsymbol{\mathcal{H}}(\mathbf{r}_2) = \oint_C \frac{I_1 d\mathbf{r}_1 \times \hat{\mathbf{r}}_{12}}{4\pi r_{12}^2} \tag{4.3}$$

4.2.4 Caution

The physical meaning of the Biot-Savart law as expressed in equation 4.2 is restricted to the notion that when it is integrated over a complete circuit, as in equation 4.3, it correctly gives the magnetic field of that circuit.

As is not possible to obtain a current element in isolation, we should be cautious in attaching a physical interpretation to equation 4.2 alone. If such an attempt is made, for example, by using the formula and the law of force (to be introduced later) to calculate the forces exerted between a pair of such current elements, it is found that the forces calculated do not obey the reaction principle, whereas the forces between two charges calculated by Coulomb's law do. On the other hand, if a calculation of the forces between two complete circuits is made, the result does obey the reaction principle.

We must therefore regard equation 4.3 as describing physical reality, while equation 4.2 can be considered as merely providing an expression for the kernel of the integral in the complete result as expressed by equation 4.3.

4.2.5 Surface and Volume Currents

As an exercise, generalise equation 4.3 to the case of surface and volume currents and obtain the results in equations 4.4 and 4.5 below.

$$\boldsymbol{\mathcal{H}}(\mathbf{r}_2) = \oint_C \frac{\boldsymbol{\mathcal{K}}(\mathbf{r}_1) \times \hat{\mathbf{r}}_{12}\, ds}{4\pi r_{12}^2} \tag{4.4}$$

$$\boldsymbol{\mathcal{H}}(\mathbf{r}_2) = \oint_C \frac{\boldsymbol{\mathcal{J}}(\mathbf{r}_1) \times \hat{\mathbf{r}}_{12}\, dv}{4\pi r_{12}^2} \tag{4.5}$$

Please note that in equation 4.4 ds is a scalar, whereas in other integrals past and to come $\mathbf{ds}$ is a vector.

4.3 Common Field Distributions

4.3.1 Long Straight Wire

Show as an exercise using the Biot-Savart law and making use of the angle α shown in Figure 4.2 that the magnetic field at a distance r from a long straight wire carrying a current I is

$$H = \frac{I}{2\pi r} \tag{4.6}$$

4.3.2 Single Turn Circular Coil

Show as an exercise that the axial field a distance z from the centre of the single-turn thin-wire circular coil shown in Figure 4.3 of radius a carrying a current I is

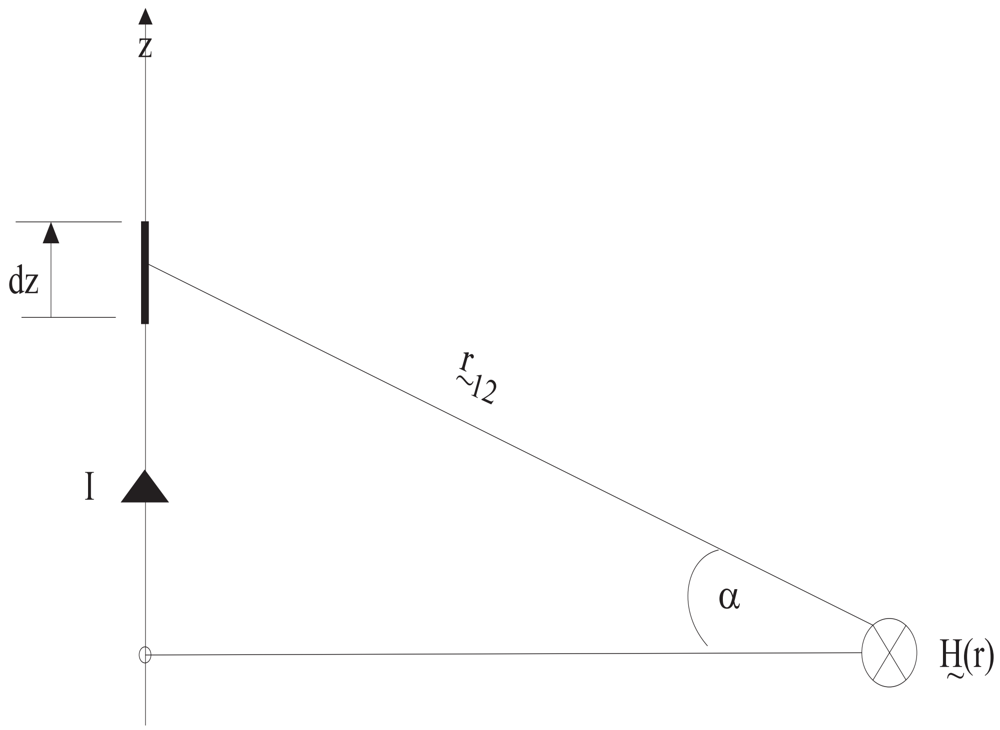


Figure 4.2: Field of a long straight wire

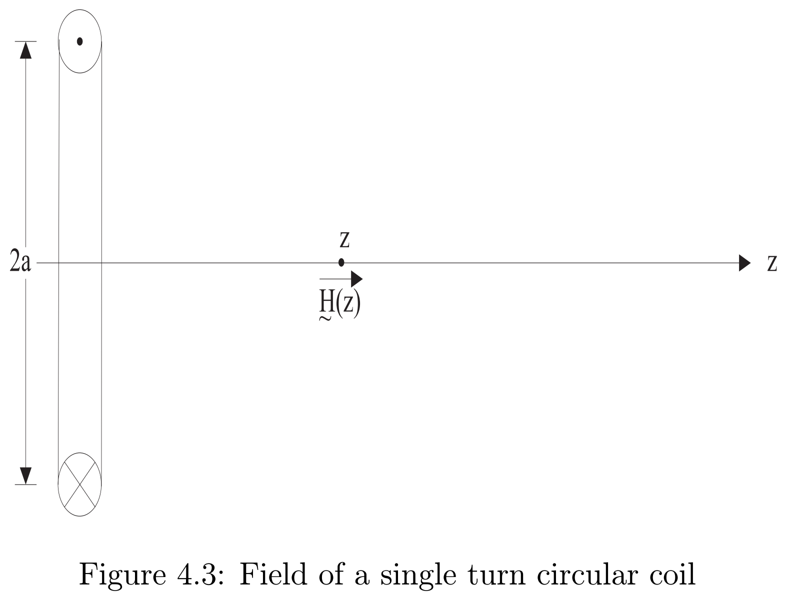


Figure 4.3: Field of a single turn circular coil

$$H_z(z) = \frac{I\, a^2}{2\,(z^2 + a^2)^{3/2}} \tag{4.7}$$

4.3.3 Field in a Toroid

The structure shown in Figure 4.4 is intended to represent a toroidal-shaped former carrying a number N closely spaced turns of wire uniformly distributed over its surface. It may be shown that the magnetic field at a radius a in the interior is given by

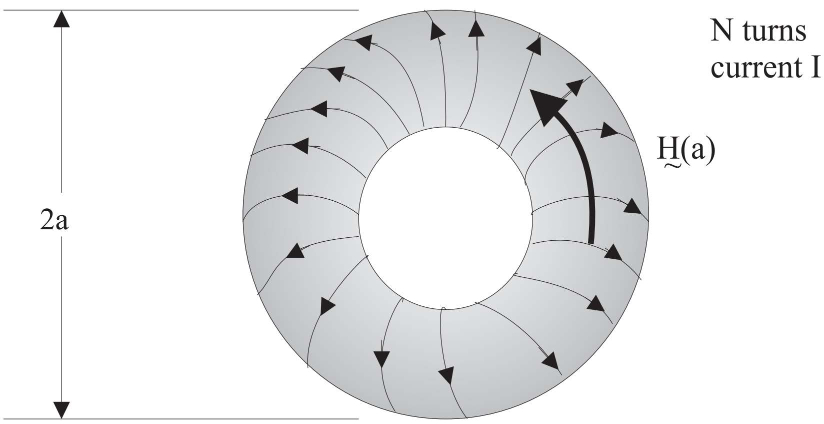


Figure 4.4: Flux-carrying Toroid

$$\boldsymbol{\mathcal{H}} = \frac{NI}{2\pi a} \tag{4.8}$$

The proof of this result proceeds very simply from Ampère's law given in Section 4.4 below.

4.3.4 Field of a Dipole

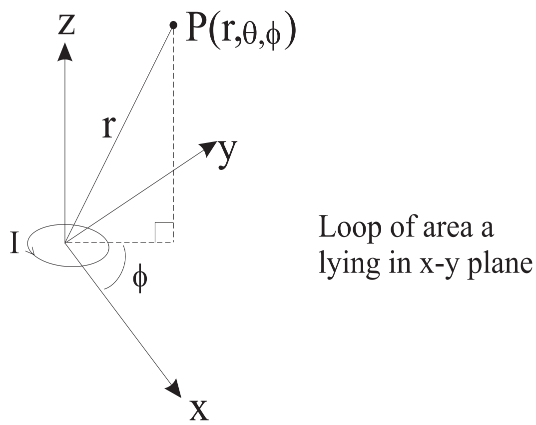


Figure 4.5: Small current carrying loop

Consider the small loop of area a lying in the xy plane in the vicinity of the origin and carrying a current I as shown in Figure 4.5. It may be shown that the field at a point $\mathbf{r}$ not near the loop is to a good approximation given by

$$H_r = \frac{2m\cos\theta}{4\pi r^3} \tag{4.9}$$

$$H_\theta = \frac{m\sin\theta}{4\pi r^3} \tag{4.10}$$

$$H_\phi = 0 \tag{4.11}$$

where

$$\mathbf{m} = I\,\mathbf{a} \tag{4.12}$$

The above result may be established by calculating either the scalar potential or the vector potential discussed in Sections 4.7 and 4.8 below, and forming the appropriate spatial derivative, but we will not produce the details here.

Because of the formal similarity between the result above and the expression for the field of an electrostatic dipole given in Section 3.6, the small current loop is known as a *magnetic dipole*. We should note however, some dissimilarities: whereas the expression for the fields of an electrostatic dipole has a factor ϵ in the denominator, no corresponding factor μ appears in the expression for the fields of a magnetic dipole.

4.4 Ampère's Circuital Law

4.4.1 Integral Form

We may show from the Biot-Savart law that if we perform the contour integral of $\boldsymbol{\mathcal{H}}$ over a closed path as shown in Figure 4.6 the result is equal to the total current enclosed, ie

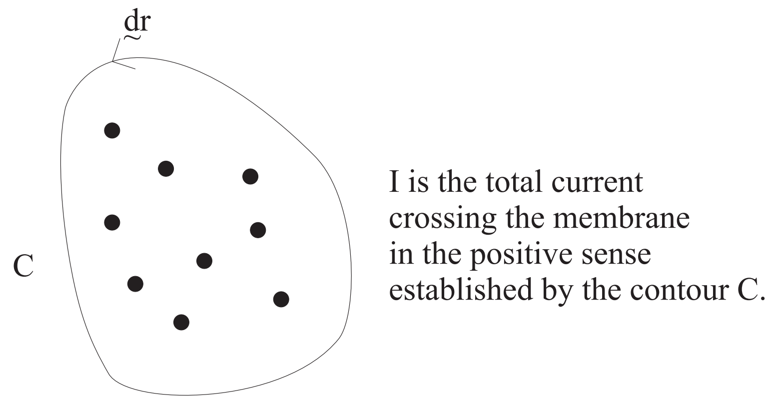


Figure 4.6: Contour integral for Amperes Law

$$\oint \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = I \tag{4.13}$$

4.4.2 Differential Form

In Figure 4.6, the dots represent the tips of vectors denoting current flow. Since

$$I = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \tag{4.14}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \tag{4.15}$$

Stokes' law will convert this to

$$\int_S (\nabla \times \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \tag{4.16}$$

.

If we let the contour C become very small so that S becomes infinitesimal we obtain the point form

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} \tag{4.17}$$

4.4.3 Notes on the Proof

The proof of Ampère's circuital law from the Biot-Savart law may be found in Section 8.7 of the book by Hayt. The proof is basically an exercise in vector calculus. It uses the magnetic vector potential which we will introduce shortly. Hayt actually proves the differential form first.

4.4.4 Beware of Substitutes

Ampère's law is often quoted in elementary courses in terms of $\boldsymbol{\mathcal{B}}$, and has μ_0 as a additional factor in the right hand side. That version of the law is correct for free space, in which $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$ are simply re-scaled versions of one another.

The $\boldsymbol{\mathcal{B}}$ version of the law becomes *incorrect* when magnetic materials are introduced, and $\boldsymbol{\mathcal{J}}$ is considered to be the conduction current (as it always should be), whereas the $\boldsymbol{\mathcal{H}}$ version of the law remains correct in the expanded context where magnetic materials are introduced.

Experience shows that students frequently fail to recognise the limited validity of the $\boldsymbol{\mathcal{B}}$ version of the law, and frequently mis-apply it, despite emphasis having been given to that limited validity. A conclusion which may be drawn is that much harm is done to students by the teaching the $\boldsymbol{\mathcal{B}}$ version, when the $\boldsymbol{\mathcal{H}}$ version could just as easily have been presented. Grump grump!

4.5 Magnetic Flux Density

4.5.1 Free Space Definition

In free space we define the magnetic flux density vector as

$$\boldsymbol{\mathcal{B}} = \mu_0 \, \boldsymbol{\mathcal{H}} \quad [\mathbf{LVE}] \tag{4.18}$$

CONTEXT	ELECTRIC	MAGNETIC
Law	Coulomb	Biot-Savart
Field	$\boldsymbol{\mathcal{E}}$	$\boldsymbol{\mathcal{H}}$
Flux density	$\boldsymbol{\mathcal{D}}$	$\boldsymbol{\mathcal{B}}$

Table 4.1: Analogies between electric and magnetic concepts

This limited validity definition will later, after magnetic materials have been described, be generalised to the form

$$\boldsymbol{\mathcal{B}} = \mu_0 \left(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}} \right) \tag{4.19}$$

The value of μ_0 is $4\pi \times 10^{-7} Hm^{-1}$. The SI units of $\boldsymbol{\mathcal{B}}$ are Wbm^{-2} or T; both are standard. Although the usage of T is very popular, use of the unit Wbm^{-2} is recommended in this course, as it directly implies the relation between magnetic flux and flux density, and also supports analogies between electric and magnetic field quantities which assist in recalling formulae.

4.5.2 Analogies

Although there are others, the preferred analogy between electric and magnetic quantities is shown in Table 4.1.

4.5.3 Magnetic Flux Φ

For a surface as shown in Figure 4.7, not necessarily closed, the magnetic flux Φ passing through the surface in its positive direction is given in equation 4.20.

$$\Phi = \int_S \boldsymbol{\mathcal{B}} \cdot \mathbf{ds} \tag{4.20}$$

This definition may be compared with the definition for electric flux Ψ in equation 4.21, in further support of the analogies presented in Section 4.5.2 above.

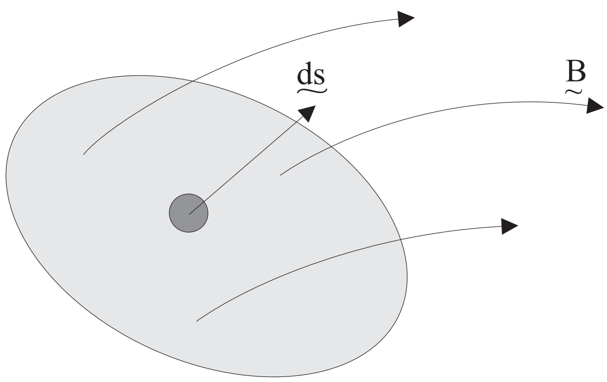


Figure 4.7: Flux intersecting a surface

$$\Psi = \int_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} \tag{4.21}$$

4.5.4 Absence of Magnetic Sources

Magnetic field and flux density clearly have vorticity, as may be seen from the two versions of Ampere's law in equation 4.22 and 4.23 below, the latter having validity restricted to free space.

$$\int_S (\nabla \times \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \tag{4.22}$$

$$\int_S (\nabla \times \boldsymbol{\mathcal{B}}) \cdot \mathbf{ds} = \mu_0 \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \quad [\mathbf{LVE}] \tag{4.23}$$

There is the question of which of them have also a divergence. The answer in free space, where only currents give rise to magnetic field and flux density, is clearly that neither has a divergence. Thus

$$\text{div}\, \boldsymbol{\mathcal{B}} = 0 \tag{4.24}$$

and

$$\oint_S \boldsymbol{\mathcal{B}} \cdot \mathbf{ds} = 0 \tag{4.25}$$

Because we are still in free space, saying that div $\boldsymbol{\mathcal{B}} = 0$ is the same as saying that div $\mu_0 \boldsymbol{\mathcal{H}} = 0$, and does not guarantee that either $\boldsymbol{\mathcal{H}}$ or $\boldsymbol{\mathcal{B}}$ will have either no divergence when media are present.

The truth is that $\boldsymbol{\mathcal{B}}$ will continue to have no divergence when media are present but $\boldsymbol{\mathcal{H}}$ will develop a divergence, so by writing div$\boldsymbol{\mathcal{B}} = 0$ we are reciting an equation which will not require amendment when we go into the more general context. To an extent, we are catering for sleepwalkers. This policy has its benefits and its dangers.

4.6 Maxwell's Equations So Far

Recalling that we are still in free space and in the static field context, we may summarise the electrostatic and magnetostatic laws so far defined as

$$\nabla \times \boldsymbol{\mathcal{E}} = 0 \quad [\mathbf{LVE}] \tag{4.26}$$
$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} \quad [\mathbf{LVE}] \tag{4.27}$$
$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho \tag{4.28}$$
$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0 \tag{4.29}$$

By choosing to write these results in terms of $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{D}}$, and $\boldsymbol{\mathcal{B}}$, we have limited the number of [**LVE**] we have had to write.

4.7 Magnetic Scalar Potential

It is possible in a limited range of contexts to derive the magnetic field $\boldsymbol{\mathcal{H}}$ and sometimes the magnetic flux density $\boldsymbol{\mathcal{B}}$ from a magnetic scalar potential, but we will not give the details in this course.

4.8 Magnetic Vector Potential

4.8.1 Definition

This potential for the magnetic density $\boldsymbol{\mathcal{B}}$ has a greater range of validity than the magnetic scalar potential, and is more commonly used. We begin with the well known equation:

$$\text{div}\,\boldsymbol{\mathcal{B}} = 0 \tag{4.30}$$

We make use of a theorem of the vector calculus that when the divergence of a vector is zero, that vector may be written as the curl of another vector, ie

$$\boldsymbol{\mathcal{B}} = \nabla \times \boldsymbol{\mathcal{A}} \tag{4.31}$$

We call $\boldsymbol{\mathcal{A}}$ the vector potential. Its units are Wbm^{-1}

4.8.2 Non-uniqueness

We must recognise that $\boldsymbol{\mathcal{A}}$ is unique to the extent of its **curl**, but it can have any divergence, as we could add to any correct $\boldsymbol{\mathcal{A}}$ a further vector field which is purely source type, and whereas the **curl** of $\boldsymbol{\mathcal{A}}$ will be unaffected, its divergence certainly will.

4.8.3 Calculation From Current Distribution

We are considering the free space case where $\boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{\mathcal{H}}$. Then

$$\boldsymbol{\mathcal{H}} = \frac{1}{\mu_0}\,\mathbf{curl}\,\boldsymbol{\mathcal{A}} \tag{4.32}$$

Hence

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} = \frac{1}{\mu_0}\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\mathcal{A}} \tag{4.33}$$

Using the vector identity **curl curl** = **grad** div $- \nabla^2$, this becomes

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} = \frac{1}{\mu_0}(\mathbf{grad}\,\text{div}\boldsymbol{\mathcal{A}} - \nabla^2\boldsymbol{\mathcal{A}}) \tag{4.34}$$

Now we have noted earlier that $\boldsymbol{\mathcal{A}}$ is not unique, and we can give it any divergence. Let us give it, in this magnetostatic case, zero divergence, and obtain

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} = -\frac{1}{\mu_0}\,\nabla^2\boldsymbol{\mathcal{A}} \tag{4.35}$$

Since **curl** $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}$ we have

$$\nabla^2 \boldsymbol{\mathcal{A}} = -\mu_0 \, \boldsymbol{\mathcal{J}} \tag{4.36}$$

Although this is a vector equation, the three Cartesian components are not inter-coupled. It is in effect three separate scalar equations in the x, y and z components of $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{J}}$.

By making comparisons between each of these scalar equations, Poissons' equation for electrostatics, and the expression for electrostatic field from charge distribution, we may write the solution immediately. The electrostatic equations we compare with are Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{4.37}$$

and its solution

$$V(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(\mathbf{r}_1)}{r_{12}} \, dv \tag{4.38}$$

In the light of these equations we see that equation 4.36 has the solution

$$\boldsymbol{\mathcal{A}}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} \int_v \frac{\boldsymbol{\mathcal{J}}(\mathbf{r}_1)}{r_{12}} \, dv \tag{4.39}$$

If the current is a filamentary current I along a contour C then equation 4.39 becomes

$$\boldsymbol{\mathcal{A}}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} \oint_C \frac{I \, \mathbf{dr}_1}{r_{12}} \tag{4.40}$$

The last two equations express a very important result which we will generalise in later years when we come to the study of electromagnetic radiation.

4.9 Magnetic Forces

4.9.1 Force on a Moving Charge

It is found experimentally that a force on a charge q moving with velocity $\mathbf{v}$ in a magnetic flux density $\boldsymbol{\mathcal{B}}$ is given by

$$\boldsymbol{\mathcal{F}} = q \, \mathbf{v} \times \boldsymbol{\mathcal{B}} \tag{4.41}$$

When an electric field is also present the force is

$$\boldsymbol{\mathcal{F}} = q \, (\boldsymbol{\mathcal{E}} + \mathbf{v} \times \boldsymbol{\mathcal{B}}) \tag{4.42}$$

The absence of odd scale factors or physical constants in this experimental relation is a consequence of the way in which the *SI* system of units for electromagnetic field quantities is constructed, with the unit of current being in effect defined by this equation, the Biot-Savart law, and the definition of μ_0 as $4\pi \times 10^{-7}$.

The result is confirmed by many varieties of electron ballistic experiments including those which are performed nightly in living rooms around the country.

4.9.2 Force on a Current Element

If we do not have a point charge q but a differential charge element $dq = \rho\, dv$ in a volume element dv, that charge moving with velocity $\mathbf{v}$, we may write (when $\boldsymbol{\mathcal{E}}$ is zero),

$$\mathbf{d}\boldsymbol{\mathcal{F}} = \rho\, dv\, \mathbf{v} \times \boldsymbol{\mathcal{B}} \tag{4.43}$$

We saw earlier that $\rho\mathbf{v}$ is just the volume current density $\boldsymbol{\mathcal{J}}$. Thus

$$\mathbf{d}\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{J}}\, dv \times \boldsymbol{\mathcal{B}} \tag{4.44}$$

Now if the volume element has small cross section, the filamentary current approximation in which $\boldsymbol{\mathcal{J}} dv$ is equivalent to $I\,\mathbf{dr}$, where $\mathbf{v}$, $\boldsymbol{\mathcal{J}}$ and $\mathbf{dr}$ are all in the same direction, may be used. Thus for a filamentary current element $I\,\mathbf{dr}$

$$\mathbf{d}\boldsymbol{\mathcal{F}} = I\,\mathbf{dr} \times \boldsymbol{\mathcal{B}} \tag{4.45}$$

4.9.3 Closed Circuit in Uniform Field

The total force $\boldsymbol{\mathcal{F}}$ on a closed circuit round a contour C carrying a current I is given by

$$\boldsymbol{\mathcal{F}} = \oint_c I\,\mathbf{dr} \times \boldsymbol{\mathcal{B}} \tag{4.46}$$

If the flux density $\boldsymbol{\mathcal{B}}$ is spatially uniform, both the current and the flux density may be taken outside the integral sign, and the resulting force is seen to be zero. If the flux density is not uniform the force need not be zero; see section 4.9.5 below.

4.9.4 Small Loop in Uniform Field

Consider the small loop shown in Figure 4.8 carrying a current I in a substantially spatially uniform magnetic flux density $\boldsymbol{\mathcal{B}}$. Since the loop is small, the flux density $\boldsymbol{\mathcal{B}}$ can be considered uniform over its extent and the total force is zero.

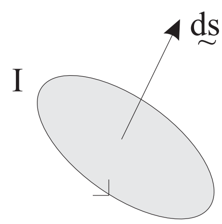


Figure 4.8: Small current carrying loop in a magnetic flux density

It is easy to show, however, that the forces on the different sections of the loop, while not producing a significant force, do produce a torque given by

$$\boldsymbol{\mathcal{T}} = I\,\mathbf{ds} \times \boldsymbol{\mathcal{B}} \tag{4.47}$$

In this relation, **ds** is the vector area of the loop, sensed in relation to the reference direction for the current, using the right hand rule. Recalling the definition for the magnetic moment $\mathbf{m} = I\,\mathbf{ds}$ of the loop, the result above becomes

$$\boldsymbol{\mathcal{T}} = \mathbf{m} \times \boldsymbol{\mathcal{B}} \tag{4.48}$$

This result may be compared with the similar result for the torque on a small electric dipole in an electric field.

$$\boldsymbol{\mathcal{T}} = \mathbf{p} \times \boldsymbol{\mathcal{E}} \tag{4.49}$$

Notice that despite the similarity, in one equation a field is involved while in the other equation a flux density is involved. This lack of symmetry between the equations is further reflected in a lack of symmetry between the units of **p** which are Cm, and **m** which are Am^2 rather than Wbm.

4.9.5 Small Dipole in Non-uniform Field

In a non-uniform flux density a small magnetic dipole can, because the forces on opposite sides no longer cancel, experience a force as well as a torque. It is easiest to analyse the case of a small rectangular loop. The result is for a loop of moment **m** a force

$$\boldsymbol{\mathcal{F}} = (\mathbf{m} \cdot \mathbf{grad})\,\boldsymbol{\mathcal{B}} \tag{4.50}$$

where

$$(\mathbf{m} \cdot \mathbf{grad}) = (m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z}) \tag{4.51}$$

is an operator we apply separately to the components B_x, B_y and B_z of $\boldsymbol{\mathcal{B}}$ to obtain the force components F_x, F_y and F_z. In detail:-

$$\begin{aligned} F_x &= m_x \frac{\partial B_x}{\partial x} + m_y \frac{\partial B_x}{\partial y} + m_z \frac{\partial B_x}{\partial z} \\ F_y &= m_x \frac{\partial B_y}{\partial x} + m_y \frac{\partial B_y}{\partial y} + m_x \frac{\partial B_y}{\partial z} \\ F_z &= m_x \frac{\partial B_z}{\partial x} + m_y \frac{\partial B_z}{\partial y} + m_z \frac{\partial B_z}{\partial z} \end{aligned} \tag{4.52}$$

4.9.6 Exercise

Prove for a small *rectangular* loop the result given in equation 4.50.

Chapter 5

ELECTRODYNAMICS

5.1 Introduction

In Chapters 3 and 4 we have developed the equations of electrostatics and magnetostatics in free space, taking into account the discoveries of Coulomb and Ampère, but not yet the contributions of Faraday or Maxwell. The equations we have developed are valid when the electric and magnetic fields are constant, and as the chapter headings imply, are in the realm of electro*statics* and magneto*statics*. In this situation there appears to be some similarity but no direct connection between the behaviour of electric and magnetic fields.

When the fields become time-varying, however, additional factors discovered first by Faraday and Maxwell come into play and, as we will see, the fields become interdependent, there being a dependence of electric field upon magnetic field, and magnetic field upon electric field. In this case we say we have entered the realm of *electrodynamics*, and the equations of previous chapters will need to be altered to take into account the new effects.

The development of theory in this chapter is purely for fields which arise from a distribution of charges and currents in free space. Consideration of both static and dynamic fields in the presence of material media is deferred until the chapter which follows.

5.2 Static Equations

We begin with a summary of the laws of electricity and magnetism in free space as they were known at about 1820, and which still apply when the fields are static. The laws will be stated first in integral form and secondly in the differential form to which they may be transformed by the application of Gauss' and Stokes' theorems.

There are four basic results. The first and third equations below state that the electrostatic field $\boldsymbol{\mathcal{E}}$ is purely source-type, and that the sources of that field are electric charge density divided by the scale factor ϵ_0. The second and fourth equations below state that the magnetic field $\boldsymbol{\mathcal{H}}$ is purely vortex-type, with the vorticity around a contour equal to the electric current enclosed.

The mathematical statement of these truths takes the form

5.2.1 Integral Form

$$\oint_C \boldsymbol{\mathcal{E}} \cdot \mathbf{dr} = 0 \qquad [\mathbf{LVE}] \tag{5.1}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.2}$$

$$\oint_S (\epsilon_0\, \boldsymbol{\mathcal{E}}) \cdot \mathbf{ds} = \int_v \rho\, dv \qquad [\mathbf{LVE}] \tag{5.3}$$

$$\oint_S (\mu_0 \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = 0 \qquad [\mathbf{LVE}] \tag{5.4}$$

In the second equation above, the contour C provides the boundary for the surface S, the senses of C and S being related by the right hand rule. In the third equation above, S is a closed surface enclosing the volume v.

The four equations above have limited validity, either through their being restricted to static fields or fields in the absence of material media.

5.2.2 Differential Form

The application of Gauss' and Stokes' theorems to these results will produce the differential forms

$$\nabla \times \boldsymbol{\mathcal{E}} = 0 \qquad [\mathbf{LVE}] \tag{5.5}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} \qquad [\mathbf{LVE}] \tag{5.6}$$

$$\nabla \cdot (\epsilon_0\, \boldsymbol{\mathcal{E}}) = \rho \qquad [\mathbf{LVE}] \tag{5.7}$$

$$\nabla \cdot (\mu_0\, \boldsymbol{\mathcal{H}}) = 0 \qquad [\mathbf{LVE}] \tag{5.8}$$

In the fourth equation above, although the factor μ_0 appears to be superfluous, it is inserted to ensure that fourth equation is, like the third equation, in terms of flux density rather than field.

As far as learning the general equations of electrodynamics is concerned, we should note that all of the equations in the two sections above are of limited validity, some because they are limited to static fields and some because of the restriction to free space. All will require amendment before becoming fully general.

As these equations are derived from the four limited validity equations of the previous section, these four equations will be of limited validity as well.

5.3 Faraday's Contribution

The discovery of electromagnetic induction by Faraday showed that when a time-varying magnetic flux is present, the electric field is no longer purely source-type, and that there can be a circulation of electric field around a contour which is equal to the negative time rate of change of magnetic flux linked by that contour. With this discovery, the laws of electrodynamics in free space appear to be

5.3.1 Integral Form

$$\oint_C \boldsymbol{\mathcal{E}} \cdot \mathbf{dr} = -\int_S \frac{\partial(\mu_0\, \boldsymbol{\mathcal{H}})}{\partial t} \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.9}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.10}$$

$$\oint_S (\epsilon_0 \boldsymbol{\mathcal{E}}) \cdot \mathbf{ds} = \int_v \rho \, dv \qquad [\mathbf{LVE}] \tag{5.11}$$

$$\oint_S (\mu_0 \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = 0 \qquad [\mathbf{LVE}] \tag{5.12}$$

5.3.2 Differential Form

The application of Gauss' and Stokes' theorem to these results will produce the differential forms

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial(\mu_0 \boldsymbol{\mathcal{H}})}{\partial t} \qquad [\mathbf{LVE}] \tag{5.13}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} \qquad [\mathbf{LVE}] \tag{5.14}$$

$$\nabla \cdot (\epsilon_0 \boldsymbol{\mathcal{E}}) = \rho \qquad [\mathbf{LVE}] \tag{5.15}$$

$$\nabla \cdot (\mu_0 \boldsymbol{\mathcal{H}}) = 0 \qquad [\mathbf{LVE}] \tag{5.16}$$

Before these equations are endorsed as the true equations of electrodynamics we must, following Maxwell, subject them to an examination for internal consistency, and discover that when all the fields are time varying, a further modification is necessary.

Again we note the limited validity of the equations of this and the previous section, the limited valididy arising partly because of the restriction of the fields to free space, and partly because the second equation in this section and in the previous section are *incorrect*, for reasons which will be discussed below.

5.4 Maxwell's Contribution

5.4.1 Conservation Inconsistency

Now that we have electromagnetic fields on both sides of the equality sign, inter-relations between the separate equations begin to appear, and we have a need to examine those equations to see that they are mutually consistent.

An important consistency condition is provided by the vector identity which states that the divergence of the curl of any vector is zero. Applying this test to the first and fourth equation, (by taking the divergence of each side of the first equation), and noting that both the time and space derivatives are partial and therefore commute, we see that the first and fourth equations are consistent.

If however we take the divergence of the second equation, we have in view of the same vector identity div **curl** $= 0$, the result

$$\nabla \cdot \boldsymbol{\mathcal{J}} = 0. \tag{5.17}$$

While this equation may be correct in the static case, it was realised by Maxwell that it is not, when the charge density is time-varying, in harmony with the charge conservation equation

$$\nabla \cdot \boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{5.18}$$

5.4.2 Displacement Current Concept

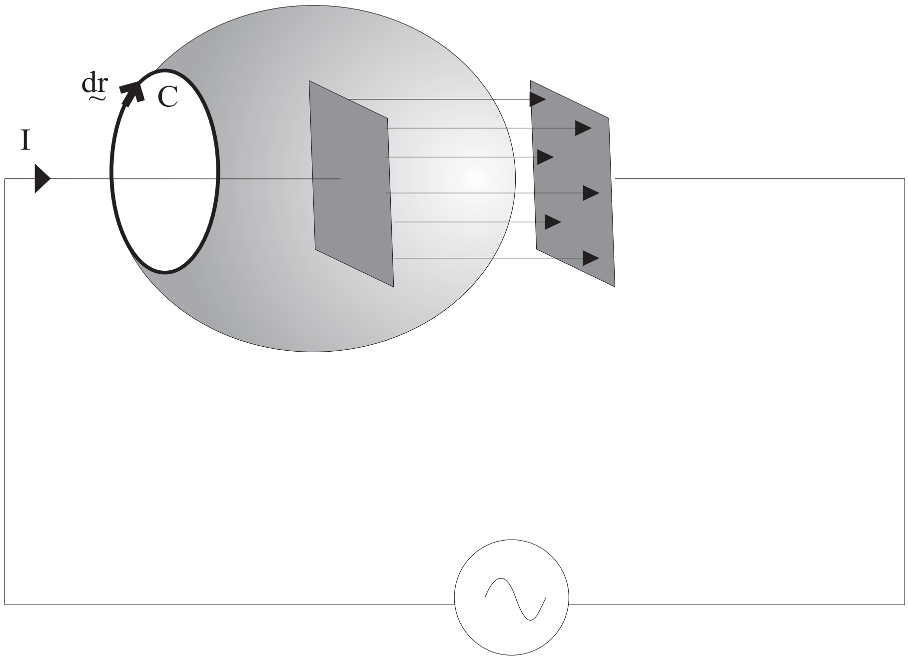


Figure 5.1: Displacement current in a capacitor

The inconsistency is made further evident by examining equation 5.10 in integral form in the light of Figure 5.1. In this figure we see an ac circuit, in which the current I in part of the circuit is carried along wires while in another part of the circuit the same current flows through the series capacitor shown.

It is possible for two different surfaces, one cutting the wire and the other passing through the capacitor plates, to be drawn such that these two surfaces have the same boundary in the form of the contour C as shown.

For each of these surfaces the surface integral on the right hand side of equation 5.10 is supposed to be equal to the line integral on the left hand side of the same equation along the common contour C. For the integral over the surface cut by the wire, the value of the integral is the current. For the integral over the surface passing between the capacitor plates there is no current, and hence the integral has a zero value.

What physically has happened is that the current has led to charges being distributed over the capacitor plates, and these charges have led to an electric flux density between the plates. This time-varying flux density effectively continues the conduction current from one plate to the other in the form of a *displacement current*

$$\frac{\partial}{\partial t} \int_S (\epsilon_0 \, \boldsymbol{\mathcal{E}}) \cdot \mathbf{ds} \tag{5.19}$$

which is just equal to the conduction current I in the wires leading to and from the capacitor.

5.4.3 Resolution

Maxwell realised that the contradiction between the differential form of Ampère's law, the charge conservation equation and the vector theorem mentioned above, and also the related paradox in the integral form just described, could both be resolved by adding the term

$$\frac{\partial(\epsilon_0\,\boldsymbol{\mathcal{E}})}{\partial t} \tag{5.20}$$

to the right hand side of equation 5.14, and making a corresponding adjustment in the integral form given by equation 5.10. The result of this action is the set of *free space* electrodynamic equations set out in the two sections below. Apart from having to take into account the effects of material media, which we will do in the next chapter, these equations have remain unchanged since their formulation by Maxwell in 1864.

5.5 Maxwell's Equations

If we now take the opportunity to collect in one place the equations as modified in this section. The result are given in the two sections below. Notice that these forms are not general but are pertaining to the free space case, hence the continued appearance of [LVE]. We are deliberately writing the equations in all ways, both in integral and differential form and also in the frequency and the time domain.

5.5.1 Differential Form

Time domain

$$\nabla\times\boldsymbol{\mathcal{E}} = -\frac{\partial(\mu_0\,\boldsymbol{\mathcal{H}})}{\partial t} \qquad \textbf{[LVE]} \tag{5.21}$$

$$\nabla\times\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial(\epsilon_0\,\boldsymbol{\mathcal{E}})}{\partial t} \qquad \textbf{[LVE]} \tag{5.22}$$

$$\nabla\cdot(\epsilon_0\,\boldsymbol{\mathcal{E}}) = \rho \qquad \textbf{[LVE]} \tag{5.23}$$

$$\nabla\cdot(\mu_0\,\boldsymbol{\mathcal{H}}) = 0 \qquad \textbf{[LVE]} \tag{5.24}$$

Frequency domain

$$\nabla\times\mathbf{E} = -j\omega\,(\mu_0\mathbf{H}) \qquad \textbf{[LVE]} \tag{5.25}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + j\omega\,(\epsilon_0\mathbf{E}) \qquad \textbf{[LVE]} \tag{5.26}$$

$$\nabla\cdot(\epsilon_0\mathbf{E}) = \rho \qquad \textbf{[LVE]} \tag{5.27}$$

$$\nabla\cdot(\mu_0\mathbf{H}) = 0 \qquad \textbf{[LVE]} \tag{5.28}$$

5.5.2 Integral Form

Time domain

$$\oint_C \boldsymbol{\mathcal{E}} \cdot \mathbf{dr} = -\int_S \frac{\partial(\mu_0 \boldsymbol{\mathcal{H}})}{\partial t} \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.29}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} + \int_S \frac{\partial(\epsilon_0 \boldsymbol{\mathcal{E}})}{\partial t} \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.30}$$

$$\oint_S (\epsilon_0 \boldsymbol{\mathcal{E}}) \cdot \mathbf{ds} = \int_v \rho \, dv \qquad [\mathbf{LVE}] \tag{5.31}$$

$$\oint_S (\mu_0 \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = 0 \qquad [\mathbf{LVE}] \tag{5.32}$$

Frequency domain

$$\oint_C \mathbf{E} \cdot \mathbf{dr} = -\int_S j\omega \, (\mu_0 \mathbf{H}) \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.33}$$

$$\oint_C \mathbf{H} \cdot \mathbf{dr} = \int_S \mathbf{J} \cdot \mathbf{ds} + \int_S j\omega \, (\epsilon_0 \mathbf{E}) \cdot \mathbf{ds} \qquad [\mathbf{LVE}] \tag{5.34}$$

$$\oint_S (\epsilon_0 \mathbf{E}) \cdot \mathbf{ds} = \int_v \rho \, dv \qquad [\mathbf{LVE}] \tag{5.35}$$

$$\oint_S (\mu_0 \mathbf{H}) \cdot \mathbf{ds} = 0 \qquad [\mathbf{LVE}] \tag{5.36}$$

Provided the charges and currents creating the fields are in free space, these equations now apply with arbitrary time variation, and will suffice, for example, for the derivation of plane waves in free space as is done in Chapter 9. The equations are not, however, sufficient for the majority of engineering applications, in which material media are frequently present. The development of fully general equations for this case is taken up in the next chapter.

Chapter 6

MATERIAL MEDIA

6.1 Introduction

In this Chapter we complete the exposition of the basic laws of electrodynamics by introducing the effects of material media. We consider first dielectric materials, and introduce the polarisation vector $\boldsymbol{\mathcal{P}}$ as a descriptor of the response of a dielectric medium to an internal electric field $\boldsymbol{\mathcal{E}}$. Then we see how Maxwell's equations are altered to take into account additional contributions to the electromagnetic field arising from the polarisation.

Next we turn our attention to magnetic materials, and introduce the magnetisation vector $\boldsymbol{\mathcal{M}}$ as a descriptor of the response of a magnetic medium to an internal magnetic field $\boldsymbol{\mathcal{H}}$, and after a brief discussion of various models of magnetic effects, we see how Maxwell's equations are altered in a model-independent way to take into account the additional contributions to the electromagnetic field associated with the magnetisation.

Attention then turns to the consideration of the properties of particular types of dielectric and magnetic media, and appropriate mathematical models and constitutive parameters are introduced to describe various types of behaviour.

Finally the capacity of internal polarisation and magnetisation to oppose any applied field is illustrated through the theory of depolarising and demagnetising factors which can be defined for linear media of ellipsoidal shape.

6.2 Properties of Dielectric Materials

We begin our study of dielectric media with a reminder of our model of an elementary dipole shown in Figure 6.1 as a pair of charges $-q$ and q which are separated by a vector distance $2\vec{\delta}$. In material media we consider that when an internal electric field $\boldsymbol{\mathcal{E}}$ is present, the positive and negative charges within an atom may become separated to create a dipole of the form shown in Figure 6.1 and which is aligned along the direction of the field. When the internal field is zero, we assume the charges are not separated and no dipoles are created. The strength $\mathbf{p}$ of each dipole is given by

$$\mathbf{p} = 2q\vec{\delta} \tag{6.1}$$

The units of $\mathbf{p}$ are Cm.

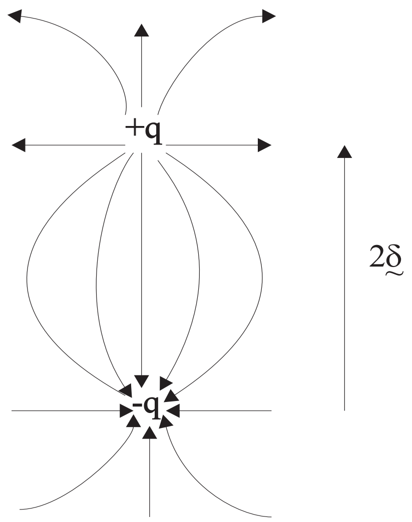


Figure 6.1: An elementary dipole

6.2.1 Polarisation Vector

If in a polarised medium there are N such dipoles each of strength $\mathbf{p}$ created per unit volume, we say the medium has a polarisation $\boldsymbol{\mathcal{P}}$ given by

$$\boldsymbol{\mathcal{P}} = N\mathbf{p} \tag{6.2}$$

The units of $\boldsymbol{\mathcal{P}}$ are Cm^{-2}.

6.2.2 Elementary Dipoles

It may be shown that in most respects a number of individual dipoles placed close together acts as a single dipole of strength equal to the sum of the strengths of the individual dipoles. In the light of this result, a small volume δv of polarised material can act as a single dipole of strength $\boldsymbol{\mathcal{P}}\delta v$.

6.2.3 Charge Crossing Plane

When a material becomes polarised positive charges move in one direction and negative charges move in the opposite direction, with the result as shown in Figure 6.2 that there is a net movement of charge across any plane in the material perpendicular to the direction of polarisation.

To determine the net charge which crosses an area A, we consider the number of polarised atoms which lie in a slice of thickness δ and area A just to the left of the plane, and also the number of polarised atoms which lie within a slice of thickness δ and area A just to the right of the plane, as each of these atoms will contribute an amount q to the movement of charge across the area A. In this way we conclude that the total charge Q which crosses the area A is given by

$$Q = (q\delta A + q\delta A)N \tag{6.3}$$

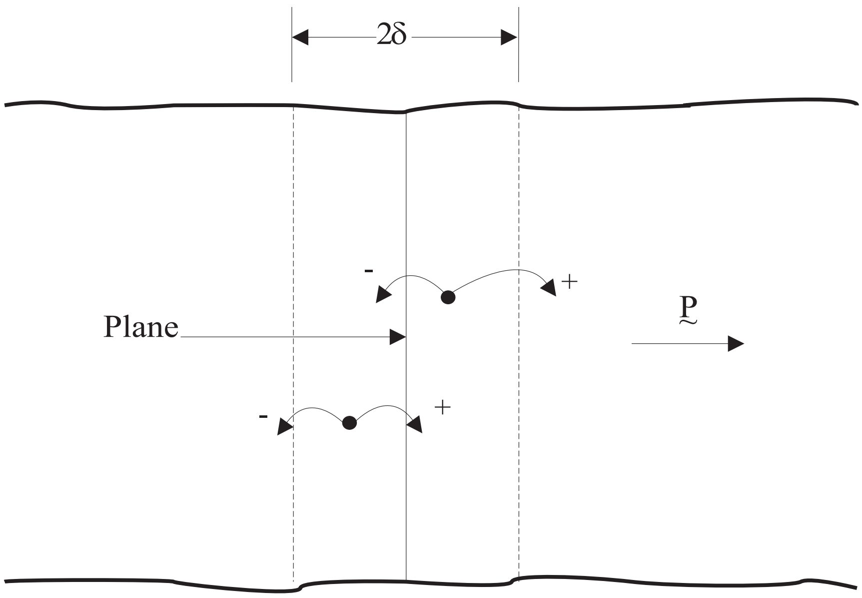


Figure 6.2: Plane in polarised dielectric

ie

$$Q = PA \tag{6.4}$$

Considering the more general case when the plane $\boldsymbol{\mathcal{A}}$ is not perpendicular to the polarisation, we arrive at the result for the charge crossing the vector area $\boldsymbol{\mathcal{A}}$ of the plane

$$Q = \boldsymbol{\mathcal{P}} \cdot \boldsymbol{\mathcal{A}} \tag{6.5}$$

6.2.4 Induced Charge Density

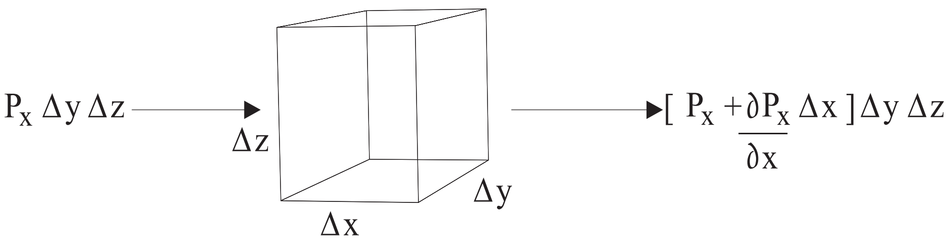


Figure 6.3: Non-uniformly polarised dielectric

The result just derived may be applied with the aid of Figure 6.3 to the calculation of what happens when a dielectric is spatially non-uniformly polarised. For the rectangular prism of sides Δx, Δy and Δz as shown, we calculate for three pairs of opposite faces the difference between the charge entering the box and the charge leaving the box, and sum the results to obtain for the nett charge entering the result

$$\left\{-\frac{\partial \mathbf{P}_x}{\partial x} - \frac{\partial \mathbf{P}_y}{\partial y} - \frac{\partial \mathbf{P}_z}{\partial z}\right\} \Delta x \Delta y \Delta z \tag{6.6}$$

Dividing by the volume of the box we find that the induced charge density per unit volume ρ_v^i is given by

$$\rho_v^i = -\frac{\partial \mathbf{P}_x}{\partial x} - \frac{\partial \mathbf{P}_y}{\partial y} - \frac{\partial \mathbf{P}_z}{\partial z} \tag{6.7}$$

Thus we see that polarisation of a medium can, if that polarisation is spatially non-uniform, create a volume charge density, even when the medium is an insulator. We use the superscript i to show that the charge density is still associated with charges which are bound to atoms and are not free to move about to form conduction currents. Using standard vector notation for the expression above we have the induced charge density per unit volume is given by

$$\rho_v^i = -\nabla \cdot \boldsymbol{\mathcal{P}} \tag{6.8}$$

6.2.5 Surface Charge Density

When the polarised medium comes to an abrupt end as shown in Figure 6.4 we consider the charge crossing a plane placed as close as we please to the end of the medium but still inside it to conclude that at the surface of the medium (or just inside it) there is an induced surface charge density

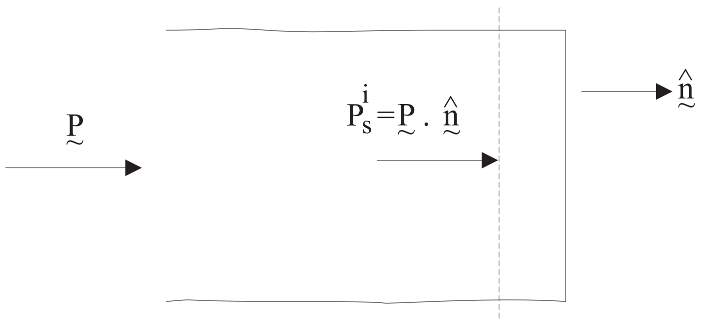


Figure 6.4: Induced charge at dielectric surface

$$\rho_s^i = \boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{n}} \tag{6.9}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface *and pointing outward.*

6.2.6 Induced Current Density

If the polarisation $\boldsymbol{\mathcal{P}}$ is changing with time, the time rate of change in the charge per unit area crossing a plane is effectively a volume current density. Thus if $\boldsymbol{\mathcal{P}}$ changes with time, we have an induced volume current density

$$\boldsymbol{\mathcal{J}}^i = \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} \tag{6.10}$$

Again we employ the superscript i to show that the electrons which participate in this induced current density are bound to the atoms and whereas they do while the polarisation is changing contribute to a current density, they are less free to move than are unbound electrons, and could not for example provide any contribution to direct current.

6.2.7 Effect on Maxwell's Equations

To see what effect these polarisation charges and currents might have on the laws of electrodynamics, we take the view that the electrons which are involved are just as effective in producing a divergence of electric field $\boldsymbol{\mathcal{E}}$ or a curl of the magnetic field $\boldsymbol{\mathcal{H}}$ as are free electrons. Thus they should be included in any charges or currents which appear on the right hand sides of the existing equations describing sources of $\boldsymbol{\mathcal{E}}$ or vortices of $\boldsymbol{\mathcal{H}}$. On the other hand, since the bound electrons participating in individual volume charge densities and currents are less free to move than are those electrons which participate in conduction currents, we may still wish to recognise them separately in equations. Thus while we retain the equation expressing Gauss' law

$$\nabla \cdot (\epsilon_0 \, \boldsymbol{\mathcal{E}}) = \rho \tag{6.11}$$

and in that equation interpret the right hand side as a *total* charge density ρ^t, it is useful to recognise that this total charge density is made up of a *conduction* charge density ρ^c and an *induced* charge density ρ^i, ie

$$\rho^t = \rho^c + \rho^i \tag{6.12}$$

Thus explicitly showing the separate components of charge density on the right hand side we have

$$\nabla \cdot (\epsilon_0 \, \boldsymbol{\mathcal{E}}) = \rho^c + \rho^i \tag{6.13}$$

But we have already shown that

$$\rho^i = -\nabla \cdot \boldsymbol{\mathcal{P}} \tag{6.14}$$

Thus

$$\nabla \cdot (\epsilon_0 \, \boldsymbol{\mathcal{E}}) = \rho^c - \nabla \cdot \boldsymbol{\mathcal{P}} \tag{6.15}$$

ie

$$\nabla \cdot (\epsilon_0 \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}}) = \rho^c \tag{6.16}$$

The advantages of writing the result in this way is that the right hand side now contains only the *conduction* charges, and it is these charges which are more directly under our control in that they flow along wires and into and out of terminals we have constructed, or are constrained to remain on electrodes which we are moving about.

Now we introduce the general definition for the electric flux density vector $\boldsymbol{\mathcal{D}}$

$$\boldsymbol{\mathcal{D}} = (\epsilon_0 \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}}) \tag{6.17}$$

In terms of $\boldsymbol{\mathcal{D}}$, Gauss' law is now written as

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho^c \tag{6.18}$$

Very often the superscript ^{c} is left off the charge density and it is assumed from the context that conduction charge density is intended to be denoted by the symbol ρ.

Now if the induced current density $\boldsymbol{\mathcal{J}}^i$ is as effective as the conduction current density $\boldsymbol{\mathcal{J}}^c$ in creating vortices of $\boldsymbol{\mathcal{H}}$, we must amend Ampère's law, (as already amended by Maxwell) from the form

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial(\epsilon_0\, \boldsymbol{\mathcal{E}})}{\partial t}\ [\mathbf{LVE}] \tag{6.19}$$

to the form

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}^c + \boldsymbol{\mathcal{J}}^i + \frac{\partial(\epsilon_0\, \boldsymbol{\mathcal{E}})}{\partial t} \tag{6.20}$$

Noting that

$$\boldsymbol{\mathcal{J}}^i = \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} \tag{6.21}$$

we have

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}^c + \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} + \frac{\partial(\epsilon_0\, \boldsymbol{\mathcal{E}})}{\partial t} \tag{6.22}$$

Making use of the general definition of $\boldsymbol{\mathcal{D}}$ as given above this equation takes the more compact form

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}^c + \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \tag{6.23}$$

Very often the superscript ^{c} is left off the $\boldsymbol{\mathcal{J}}^c$ and it is assumed from the context that conduction current is intended to be denoted by the symbol $\boldsymbol{\mathcal{J}}$.

6.3 Properties of Magnetic Materials

The two most evident properties of magnetic materials are firstly their capacity to form *permanent magnets* which can, without the benefit of any detectable internal electric current, both create magnetic fields and themselves experience forces and torques in magnetic fields, and secondly their capacity to enhance a magnetic flux density which may have been created, in the region occupied by the magnetic material, by currents or other magnets.

In the investigation of both of these phenomena it becomes evident that magnetised material exhibits properties mathematically similar to those of electrically polarised material, in that the fields created are of dipolar form. There is however no capacity to form fields of a unipolar form, such as can be created in electrostatics by isolated electrons.

In developing macroscopic models for magnetic effects we have a choice between modelling the magnetic dipoles which appear to exist within magnetic material as having been caused by *amperian currents* or alternatively by *magnetic charges*. In the light of this fact we will review briefly below the properties of those two separate models for a magnetic dipole.

6.3.1 Magnetic Dipole: Current Model

The electric current model of a dipole is shown in Figure 6.5. In the figure, the current I circulating around a small loop of area $\mathbf{a}$ directed along the z axis produces the dipolar field

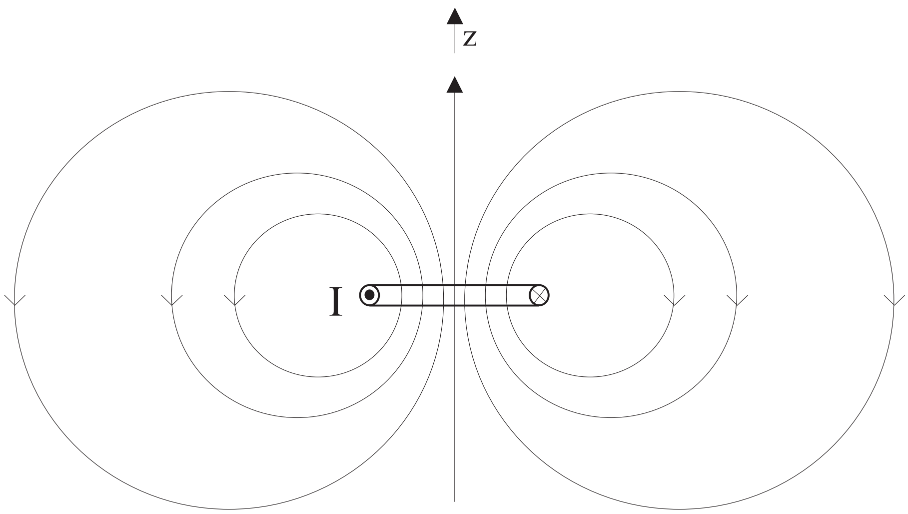


Figure 6.5: Current model of a dipole

$$H_r = \frac{2m\cos\theta}{4\pi r^3} \tag{6.24}$$

$$H_\theta = \frac{m\sin\theta}{4\pi r^3} \tag{6.25}$$

$$H_\phi = 0 \tag{6.26}$$

where

$$\mathbf{m} = I\,\mathbf{a} \tag{6.27}$$

is defined as the strength of the dipole in units of Am^2. In a magnetic flux density of $\boldsymbol{\mathcal{B}}$, the dipole will experience a torque given by

$$\boldsymbol{\mathcal{T}} = \mathbf{m} \times \boldsymbol{\mathcal{B}} \tag{6.28}$$

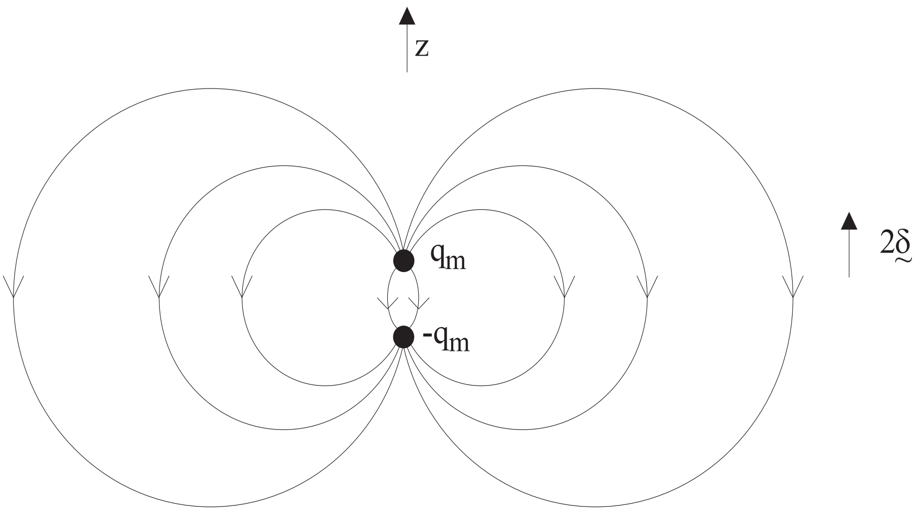


Figure 6.6: Magnetic charge model of a dipole

6.3.2 Magnetic Dipole: Magnetic Charge Model

The magnetic charge model of a dipole is shown in Figure 6.6. In the figure, magnetic charges of q_m and $-q_m$ which are separated by a vector distance of $2\vec{\delta}$ directed along the z axis again produce the dipolar field

$$H_r = \frac{2m\cos\theta}{4\pi r^3} \tag{6.29}$$

$$H_\theta = \frac{m\sin\theta}{4\pi r^3} \tag{6.30}$$

$$H_\phi = 0 \tag{6.31}$$

where

$$\mathbf{m} = 2q_m\vec{\delta}/\mu_0 \tag{6.32}$$

is defined as the strength of the dipole. In this definition, the units of magnetic charge q_m are Wb, while the units of dipole strength are as before Am^2. In a magnetic flux density of $\boldsymbol{\mathcal{B}}$ the dipole will experience the same torque as given in equation 6.28.

6.3.3 Notes on Units

In defining the strength of the dipole we have an asymmetry between the magnetic and electric dipole cases. The electric dipole strength is defined as $2q\vec{\delta}$ without involving ϵ_0, and has units of Cm. The magnetic dipole strength is defined as $2q\vec{\delta}/\mu_0$ ie with a μ_0 in the denominator, and has units of Am^2 . This asymmetry also appears in the expressions given in this and earlier chapters for the torques on electric and magnetic dipoles, and also in the relations between $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{D}}$ and the not quite conformal relations between $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{B}}$.

It is reasonable to regard all of these asymmetries as arising from the way in which magnetisation in the SI system of units was, after many debates within international committees in the middle part of this century, eventually defined. Prior to the adoption of the SI system, competing systems of units with the alternative relations $\boldsymbol{\mathcal{B}} = \mu_0 (\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}})$, $\boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{B}} = \mu_0 (\boldsymbol{\mathcal{H}} + 4\pi \boldsymbol{\mathcal{M}})$ and $\boldsymbol{\mathcal{B}} = \mu_0 \boldsymbol{\mathcal{H}} + 4\pi \boldsymbol{\mathcal{M}}$, were in use. Although the relation eventually chosen may be seen by some as being less than optimum, we can at least be pleased that the confusion of having four conflicting conventions has almost disappeared from current literature.

6.3.4 Magnetic Materials

Now that we have a description of a magnetic medium (in fact we have two of them) we must see whether the chosen model can be incorporated into the laws of electrodynamics so that both that description and the modified laws are in accord with experiment. In this process we must be guided by the over-riding concern of empirical validity — it is the agreement with the results of experiment which will determine the acceptability of any model.

But we can also be guided by the consideration of convenience. Where two equivalent descriptions which produce the same results at the level at which we can make observations are available, we are at liberty to choose, if we wish, the one which is based on the simpler conceptual scheme. We are also at liberty to make a choice on the matter of taste — one model may be more appealing than another. But having made a choice on whatever grounds, we should continue to be aware that a choice has been made, and not be tempted to press the model beyond the boundaries of our ability to perform experimental validation.

In the modelling of magnetic media we have a choice and a temptation of the nature just described. Our experimental evidence is that atoms of magnetic media behave as infinitesimal magnetic dipoles ie points which can generate externally to themselves a magnetic field $\boldsymbol{\mathcal{H}}$ which is of the same shape as the electric field $\boldsymbol{\mathcal{E}}$ of the electric dipole studied in Section 3.6, which has already been shown to be of the same shape as the field $\boldsymbol{\mathcal{H}}$ of the infinitesimal current loop studied in Section 4.3.4.

Thus we have a choice between modelling the magnetic moment of an atom as having been caused either by a small current loop or by a pair of separated magnetic charges of opposite sign. The choice must be made on the grounds of convenience or of taste, because we are constructing a macroscopic theory which does not concern itself with the processes which occur within the atom, but only with the exterior fields.

Whatever our model of the individual dipoles of strength $\mathbf{m}$, when there are N dipoles present per cubic metre, we define the magnetisation $\boldsymbol{\mathcal{M}}$ as

$$\boldsymbol{\mathcal{M}} = N\,\mathbf{m} \tag{6.33}$$

Before we discuss further aspects of models of magnetisation, we must examine a property of the uniformly magnetised sphere shown in Figure 6.7, that the magnetisation vector $\boldsymbol{\mathcal{M}}$ has on the surface both a divergence and a curl, in fact an infinite value for each, if we are prepared to accept such singularities. Thus if we are going to incorporate $\boldsymbol{\mathcal{M}}$ into our electrodynamic model through an equation of the form $\boldsymbol{\mathcal{B}} = \mu_0(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}})$

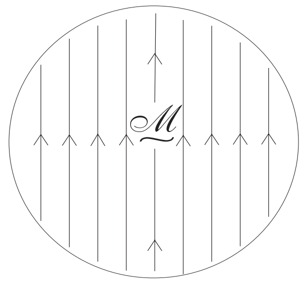


Figure 6.7: A uniformly magnetised sphere

analogous to the relation $\boldsymbol{\mathcal{D}} = \epsilon_0 \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}}$, (which is what we in fact do) some adjustment of the divergence and curl of either or both of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}$ will be needed.

In the amperian current model we imagine that the magnetic dipole moment of an atom is caused by a miniature circulating current, and that when an aggregate of such atoms is present in a sample of uniformly magnetised material, these currents effectively cancel in the interior of the material, but do not cancel at the surface where they form a skin of surface current density. We focus our attention on these imaginary amperian currents as the cause of magnetic properties of matter.

These amperian currents are not supposed to make a contribution to the magnetic field $\boldsymbol{\mathcal{H}}$, of which the vortices are considered by definition only to arise from conduction currents, but are supposed to make, under assumptions made in model, a contribution to the magnetic flux density $\boldsymbol{\mathcal{B}}$, giving it additional vortices, but of course no sources. As has been explained above, as $\boldsymbol{\mathcal{B}}$ is given only vortices, any sources present in $\boldsymbol{\mathcal{M}}$ must be given (albeit with a sign change) to $\boldsymbol{\mathcal{H}}$. In the amperian current model, however, there is so little discussion of $\boldsymbol{\mathcal{H}}$ that this fact generally passes without notice.

In the magnetic charge model, we imagine that the magnetic dipole moment of an atom is caused by a separation within the atom of bound magnetic charges of opposite sign. When an aggregate of such atoms is present to form a sample of uniformly magnetised material, the infinitesimal motion of positive magnetic charge toward one surface produces a surface density of north magnetic pole strength, while the infinitesimal movement of negative magnetic charge toward the opposite surface produces there a surface density of south magnetic pole strength. These magnetic poles are supposed by definition to act as sources of magnetic field $\boldsymbol{\mathcal{H}}$ which is now seen as a vector which has both sources and vortices, the sources being the abovementioned magnetic poles, while the vortices are the conduction currents as described in Chapter 4.

There is considered to be no change to the vorticity of $\boldsymbol{\mathcal{H}}$ caused by amperian currents as these creatures do not exist in the magnetic charge model. Because, however, $\boldsymbol{\mathcal{M}}$ has as discussed earlier its own vorticity, and we are making no change in the vorticity of $\boldsymbol{\mathcal{H}}$, there will be an additional contribution to the vorticity of $\boldsymbol{\mathcal{B}}$, although in the magnetic charge model this fact is not made as prominent as is the introduction of sources for $\boldsymbol{\mathcal{H}}$.

6.3.5 Definition of Magnetic Flux Density

With the above definition as background, and whatever be our explanation for the magnetic properties of atoms, we now formally modify the previously given definition of the magnetic flux density vector $\boldsymbol{\mathcal{B}}$ to the form

$$\boldsymbol{\mathcal{B}} = \mu_0 \left(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{M}} \right) \tag{6.34}$$

We should observe firstly the similarity to the definition of the electric flux density vector $\boldsymbol{\mathcal{D}}$

$$\boldsymbol{\mathcal{D}} = \epsilon_0 \, \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}} \tag{6.35}$$

and should also note the difference that μ_0 is a co-efficient of $\boldsymbol{\mathcal{M}}$ in equation 6.34, whereas ϵ_0 is not a co-efficient of $\boldsymbol{\mathcal{P}}$ in equation 6.35.

6.3.6 Magnetostatic Effects

Having made this modification, we note first that whichever model of magnetisation is used, we will still have $\nabla \cdot \boldsymbol{\mathcal{B}} = 0$. This is because:

- In the amperian current model we have the flux density $\boldsymbol{\mathcal{B}}$ caused by some combination of conduction currents and amperian currents, and either a field or a flux density caused by a current forms only closed loops.
- In the magnetic charge model we have the magnetic field $\boldsymbol{\mathcal{H}}$ possessing a vortex component caused by conduction currents as expressed by the Biot-Savart law (or its equivalent in Ampere's law), and $\boldsymbol{\mathcal{H}}$ possessing also a source-type contribution from the magnetic charges. These magnetic charges will produce a pole strength per unit area at the ends of a bar magnet where the material ends, and a pole strength per unit volume in the interior of the material when the magnetisation is non-uniform, just as in dielectrics where there is an induced charge density per unit area where the dielectric material ends, and an induced charge per unit volume when the polarisation is non-uniform.

In this model we will have $\boldsymbol{\mathcal{H}}$ flowing out from such regions of pole strength as is shown in Figure 6.8. It is important to note that this diagram shows only the component of $\boldsymbol{\mathcal{H}}$ which arises from the magnetic poles caused by the discontinuity of $\boldsymbol{\mathcal{M}}$; in general there would be other components of $\boldsymbol{\mathcal{H}}$ caused by currents or poles elsewhere.

Thus in both formulations we have arranged that $\boldsymbol{\mathcal{B}}$ is a source-free flux density, and that $\boldsymbol{\mathcal{H}}$ has both sources and vortices as shown in the equations below.

For $\boldsymbol{\mathcal{B}}$ we have vortices but no sources, ie

$$\nabla \times \boldsymbol{\mathcal{B}} = \mu_0 \left(\boldsymbol{\mathcal{J}}^c + \nabla \times \boldsymbol{\mathcal{M}} \right) \tag{6.36}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0 \tag{6.37}$$

while for $\boldsymbol{\mathcal{H}}$ we have both vortices and sources, ie

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}^c \tag{6.38}$$

$$\nabla \cdot \boldsymbol{\mathcal{H}} = -\nabla \cdot \boldsymbol{\mathcal{M}} \tag{6.39}$$

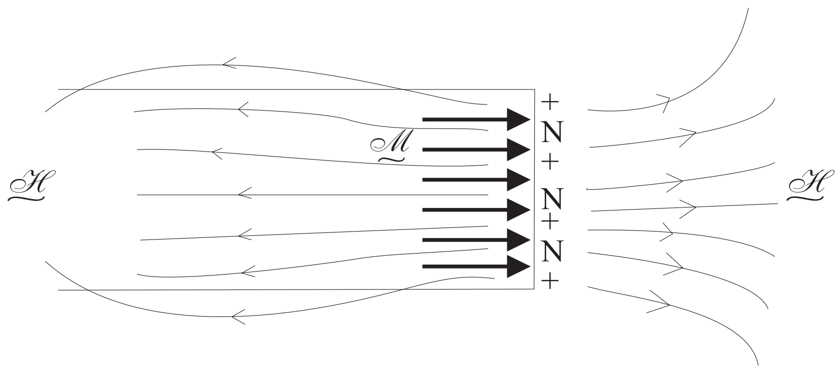


Figure 6.8: Magnetic pole at end of a bar magnet

Both the above sets of equations are true in all contexts (provided the fields are static) and in both the amperian current and the magnetic charge model of magnetisation.

If we wish to adopt the amperian current model, and wish to do so in a very overt form, we can call the second term in the brackets on the right hand side of equation 6.36 the amperian current density $\boldsymbol{\mathcal{J}}^m$ and write

$$\nabla \times \boldsymbol{\mathcal{B}} = \mu_0 \left(\boldsymbol{\mathcal{J}}^c + \boldsymbol{\mathcal{J}}^m \right) \tag{6.40}$$

If we wish to adopt the magnetic charge model, and wish to do so in a very overt form, we can represent the right hand side of equation 6.39 in terms of an induced magnetic charge density ρ_m^i and the scale factor μ_0 and write

$$\nabla \cdot \boldsymbol{\mathcal{H}} = -\rho_m^i / \mu_0 \tag{6.41}$$

Most engineers work with equations 6.36 to 6.39 without overtly adopting either model as is done in equation 6.40 or equation 6.41. We note that as the fields are still static, we have still not produced fully general versions of Maxwell's equations.

6.4 Full Electrodynamic Equations

We now produce complete equations of electrodynamics by introducing the following experimental observations:

- As Faraday has discovered, a time rate of change of magnetisation $\boldsymbol{\mathcal{M}}$ is as effective as a time rate of change of a free space magnetic flux density $\mu_0 \boldsymbol{\mathcal{H}}$ in inducing an electric field, ie it is the time rate of change of the total magnetic flux density $\boldsymbol{\mathcal{B}}$ which counts in this matter.

- The polarisation current $\boldsymbol{\mathcal{J}}^p$ is as effective as the conduction current $\boldsymbol{\mathcal{J}}^c$ in creating a vortex of magnetic field $\boldsymbol{\mathcal{H}}$.

This last result is not too surprising, as we explain both dielectric polarisation and electronic conduction as involving electrons, and the subdivision of total charge and current densities into conduction charge and current densities and polarisation charge and current densities is one which has been made as a matter of convenience. Thus the **curl** equation for $\boldsymbol{\mathcal{H}}$ in the presence of both types of current density is modified to

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial(\epsilon_0\, \boldsymbol{\mathcal{E}})}{\partial t} + \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} \tag{6.42}$$

ie

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \tag{6.43}$$

where in both of the above equations, $\boldsymbol{\mathcal{J}}$ is understood by convention to refer to conduction current density alone.

In accord with Faraday's observation, the **curl** equation for $\boldsymbol{\mathcal{E}}$ is modified to

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial(\mu_0\, \boldsymbol{\mathcal{H}})}{\partial t} - \frac{\partial(\mu_0\, \boldsymbol{\mathcal{M}})}{\partial t} \tag{6.44}$$

ie

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{6.45}$$

It might be noted that, once the divergence equations for $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ were modified as discussed earlier, modifications as indicated to the **curl** equations are also necessary to keep the full set of equations consistent with the vector theorem div **curl** $= 0$.

If we now collect in one place the equations as modified in this section we have the final, fully general, equations of electrodynamics as given in the two sections below.

6.4.1 Differential Form

Time domain

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{6.46}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \tag{6.47}$$

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho \tag{6.48}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0 \tag{6.49}$$

Frequency domain

In the sinusoidal steady state where we represent the field quantities by complex phasors the differential equations take the form

$$\nabla \times \mathbf{E} = -j\omega\, \mathbf{B} \tag{6.50}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\, \mathbf{D} \tag{6.51}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{6.52}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.53}$$

6.4.2 Integral Form

The above equations are as indicated in differential form. For completeness we quote below the same equations in the integral form to which they may be transformed through the application of Gauss' and Stokes' theorems.

Time domain

$$\oint_C \boldsymbol{\mathcal{E}} \cdot \mathbf{dr} = -\int_S \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \cdot \mathbf{ds} \tag{6.54}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} + \int_S \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \cdot \mathbf{ds} \tag{6.55}$$

$$\oint_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = \int_v \rho \, dv \tag{6.56}$$

$$\oint_S \boldsymbol{\mathcal{B}} \cdot \mathbf{ds} = 0 \tag{6.57}$$

Frequency domain

In the sinusoidal steady state where we represent the field quantities by complex phasors the integral equations take the form

$$\oint_C \mathbf{E} \cdot \mathbf{dr} = -\int_S j\omega \, \mathbf{B} \cdot \mathbf{ds} \tag{6.58}$$

$$\oint_C \mathbf{H} \cdot \mathbf{dr} = \int_S \mathbf{J} \cdot \mathbf{ds} + \int_S j\omega \, \mathbf{D} \cdot \mathbf{ds} \tag{6.59}$$

$$\oint_S \mathbf{D} \cdot \mathbf{ds} = \int_v \rho \, dv \tag{6.60}$$

$$\oint_S \mathbf{B} \cdot \mathbf{ds} = 0 \tag{6.61}$$

The significance of the different symbols for field vectors in the time domain and frequency domain must be remembered. In a time domain equation the symbol $\boldsymbol{\mathcal{E}}$ represents a vector which is real and a function of time, while $\mathbf{E}$ in a frequency domain equation represents a vector which is complex and has no time variation.

As a further point of clarification we take the opportunity to remind ourselves that the symbols ρ and $\boldsymbol{\mathcal{J}}$ or $\mathbf{J}$ appearing in all four groups of equations above are understood to refer to conduction charge and current density even though they do not contain an explicit indication of this fact.

The above four sets of equations are in *medium-independent form,* ie whatever the relations between $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{E}}$, and between $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{H}}$, these equations will apply. Now that we have the general medium-independent equations of electrodynamics, we must consider in the next two sections the behaviour of various particular types of dielectric and magnetic media, in which the polarisation and magnetisation respond to internal fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ in various specific ways.

6.5 Properties of Dielectric Media

6.5.1 General Remarks

We note first that $\boldsymbol{\mathcal{P}}$ may be, and in important cases is, a non-linear function of $\boldsymbol{\mathcal{E}}$. However, in other important cases a linear relation is a reasonable approximation to reality. We set out the notation which pertains in the linear case below, as well as describing the non-linear cases later.

6.5.2 Ideal Dielectric

In the ideal dielectric the polarisation is always linearly proportional to and in the direction of the instantaneous value of the electric field, ie

$$\boldsymbol{\mathcal{P}} = \chi_e \, \epsilon_0 \, \boldsymbol{\mathcal{E}} \tag{6.62}$$

where χ_e is a dimensionless constant called the *dielectric susceptibility.* In this equation and in the many to follow, we must understand that it is the electric field *inside* the medium which appears in the $\boldsymbol{\mathcal{P}}$ - $\boldsymbol{\mathcal{E}}$ relation, and that the field inside the medium is generally different from the field outside the medium, and that both are different from the field which pertains in the absence of the medium. Fortunately we will be able to assemble sufficiently many equations to find all of these separate fields without too much difficulty.

From the above equation it follows that the electric flux density $\boldsymbol{\mathcal{D}}$ is given by

$$\boldsymbol{\mathcal{D}} = (1 + \chi_e) \, \epsilon_0 \, \boldsymbol{\mathcal{E}} \tag{6.63}$$

An alternative expression of this result is

$$\boldsymbol{\mathcal{D}} = \epsilon_r \, \epsilon_0 \, \boldsymbol{\mathcal{E}} \tag{6.64}$$

where ϵ_r is a further dimensionless parameter known as the *relative dielectric constant.* Alternatively we may write

$$\boldsymbol{\mathcal{D}} = \epsilon \, \boldsymbol{\mathcal{E}} \tag{6.65}$$

where ϵ is the dielectric permittivity and has the units of Fm^{-1}. It may be shown that with this type of dielectric behaviour, no losses are involved in changing the state of polarisation of the material. Models of dielectric media suitable for a wide variety of purposes can be obtained by combining the above lossless $\boldsymbol{\mathcal{D}}$ - $\boldsymbol{\mathcal{E}}$ relation with the linear conductivity relation which does involve energy loss, defined in Section 6.6 below.

6.5.3 Non-linear but Lossless Dielectric

A more general form of behaviour, which admits of some non-linearity in the material, but still does not involve energy loss, is described by the equation

$$\boldsymbol{\mathcal{P}} = \chi_e(E) \, \epsilon_0 \, \boldsymbol{\mathcal{E}} \tag{6.66}$$

where $\chi_e(E)$ is now a dimensionless dielectric susceptibility which depends on the magnitude of the electric field strength. This model is of theoretical use in establishing certain energy conservation theorems, but is not much needed for description of practical materials as dielectrics are generally highly linear until breakdown is reached.

6.5.4 Linear Crystalline Dielectric

If a dielectric medium is in the form of a single crystal, it will generally exhibit different polarisation responses to electric fields in different directions. This behaviour is described by the equation

$$\boldsymbol{\mathcal{P}} = \epsilon_0 \underset{\approx}{\chi} \cdot \boldsymbol{\mathcal{E}} \tag{6.67}$$

where $\underset{\approx}{\chi}$ is called the dimensionless dielectric susceptibility tensor, whose components are specified by a 3×3 array. This model is useful in that it does represent the behaviour of practical single-crystal dielectric materials. We will not in later sections of this chapter introduce a corresponding model for single-crystal magnetic media, because inter-atomic interactions in magnetic media are quite different from those in dielectric media, with the result that single-crystal magnetic media behave in the main in an entirely different way.

6.5.5 Permanently Polarised Ferroelectric

In some dielectric media the atoms possess permanent dipole moments which can, once permanently aligned by the application of a strong electric field, remain in that condition unless other strong electric fields are applied. This behaviour is described by the equation

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_0 \tag{6.68}$$

where $\boldsymbol{\mathcal{P}}_0$ is as constant vector.

6.5.6 Linear Lossy Dielectric

In some dielectric materials, losses involved in changing the state of polarisation cannot be entirely neglected. In these materials, we can sometimes write a linear differential equation relating the polarisation and its time rate of change, and the corresponding quantities for the electric field. If we know details of time variation, calculation of the amount of energy lost is then possible.

When the time variation is sinusoidal, the results admit of a simple description. For a sinusoidally varying electric field, the resulting polarisation response is also sinusoidal, but lags a little behind the electric field. This behaviour is described by the phasor equation

$$\mathbf{P} = (\chi_e^{'} - j\chi_e^{''})\, \epsilon_0\, \mathbf{E} \tag{6.69}$$

where χ_e' and χ_e'' are the energy-storage and energy-loss components of a complex dielectric susceptibility. Please note that although losses are involved, the response is still linear in that the phasor $\mathbf{P}$ is still proportional to the phasor $\mathbf{E}$. The corresponding description in terms of a complex dielectric permittivity is

$$\mathbf{D} = (\epsilon^{'} - j\epsilon^{''})\,\mathbf{E} \tag{6.70}$$

It might be noted, by comparing the form of this equation with that of the electrical conductivity equation appearing is Section 6.6, that although the physical mechanisms might be regarded as being different, the effects of electronic conduction and polarisation loss are, on a macroscopic scale and with sinusoidal excitation, indistinguishable. That this is true may be seen by noting that in Maxwell's equations both conduction current and displacement current fulfil the same function of providing vortices for $\boldsymbol{\mathcal{H}}$, and that if we calculate in the sinusoidal steady state case the displacement current by taking the time derivative of equation 6.70 above, we find that there is a component of displacement current proportional to and in phase with the electric field, and in these two respects has the same behaviour as a conduction current derivable from the linear conductivity relation discussed below.

6.6 Electrical Conductivity

In a medium exhibiting a linear electrical conductivity relation we expect the volume current density $\boldsymbol{\mathcal{J}}$ to be proportional to the internal electric field $\boldsymbol{\mathcal{E}}$. In field terms then, Ohm's law has the form

$$\boldsymbol{\mathcal{J}} = \sigma\,\boldsymbol{\mathcal{E}} \tag{6.71}$$

where σ denotes the *electrical conductivity*, the inverse of the electrical resistivity ρ. The units of conductivity are Sm^{-1}.

6.7 Properties of Magnetic Media

6.7.1 Types of Magnetic Effect

Although we have adopted for a magnetised medium a simple macroscopic model, in the form of a distribution of elementary dipoles per unit volume, the magnetic effects shown by material media are on the microscopic scale rather varied. The five different types of behaviour normally identified are known as diamagnetic, paramagnetic, ferromagnetic, ferrimagnetic and anti-ferromagnetic.

Diamagnetic, paramagnetic and anti-ferromagnetic materials exhibit almost no magnetisation, and have few engineering uses. The materials which show significant magnetic effects of interest to engineers are the *ferromagnetic* materials and their close relatives the *ferrimagnetic* materials. These media contain iron, cobalt or nickel, which are atoms with incomplete electronic d shells and which possess a fixed magnetic moment per atom.

Although a short-range interaction, known as the *exchange* interaction and of quantum mechanical origin, between nearest neighbours in the crystal lattice tends to keep the magnetic moments of nearest neighbour atoms aligned, this ordering does not extend over regions more than approximately a micrometre in size, so that the material in its unmagnetised state is broken up into a series of *domains,* within which the magnetic moments of atoms are aligned, but among which the moments are randomly distributed in direction. On a macroscopic scale therefore, the material appears to be unmagnetised.

The reason why the materials just mentioned can develop significant magnetic effect is that under the influence of a sufficiently strong internal magnetic field $\boldsymbol{\mathcal{H}}$ the domain walls can move to permit the growth of domains magnetised in the direction of $\boldsymbol{\mathcal{H}}$ at the expense of domains magnetised in other directions, so that the macroscopically observable magnetisation grows until eventually it reaches the limit which occurs when the material has become a single domain. After that has occurred, as each atom has only a fixed magnetic moment, and number of atoms per unit volume is also fixed, further increase in internal magnetic field does not produce an increase in magnetisation.

The various types of behaviour modelled below derive from the consideration of whether domain walls move readily or with difficulty in response to changes in internal magnetic field $\boldsymbol{\mathcal{H}}$.

6.7.2 Linear Lossless Material

In polycrystalline ferromagnets and for small values of internal field strength we may write to a reasonable level of approximation

$$\boldsymbol{\mathcal{M}} = \chi_m \, \boldsymbol{\mathcal{H}} \tag{6.72}$$

where χ_m is a dimensionless parameter called the magnetic susceptibility. The same relation may be expressed in the alternative form

$$\boldsymbol{\mathcal{B}} = \mu_0(1 + \chi_m) \, \boldsymbol{\mathcal{H}} \tag{6.73}$$

Introducing the relative magnetic permeability $\mu_r = (1 + \chi_m)$ we may write

$$\boldsymbol{\mathcal{B}} = \mu_r \mu_0 \, \boldsymbol{\mathcal{H}} \tag{6.74}$$

ie

$$\boldsymbol{\mathcal{B}} = \mu \, \boldsymbol{\mathcal{H}} \tag{6.75}$$

where μ is called the magnetic permeability and is of units Hm^{-1}.

6.7.3 Non-linear but Lossless Ferromagnet

For larger field strengths, the behaviour of a ferromagnet is better described by the hysteresis curve shown in Figure 6.9. In relation to that curve we take the opportunity to define in the figure the important concepts of *saturation magnetisation* M_0 and *co-ercive force* H_c.

If the hysteresis loop is relatively narrow, and we are prepared to accept a rather crude approximation, we could write

$$\boldsymbol{\mathcal{M}} = \chi_m(H) \, \boldsymbol{\mathcal{H}} \tag{6.76}$$

where $\chi_m(H)$ is the field-dependent non-linear magnetic susceptibility. This model corresponds to the dielectric model of equation 6.66, and has approximately the same relatively uninteresting uses.

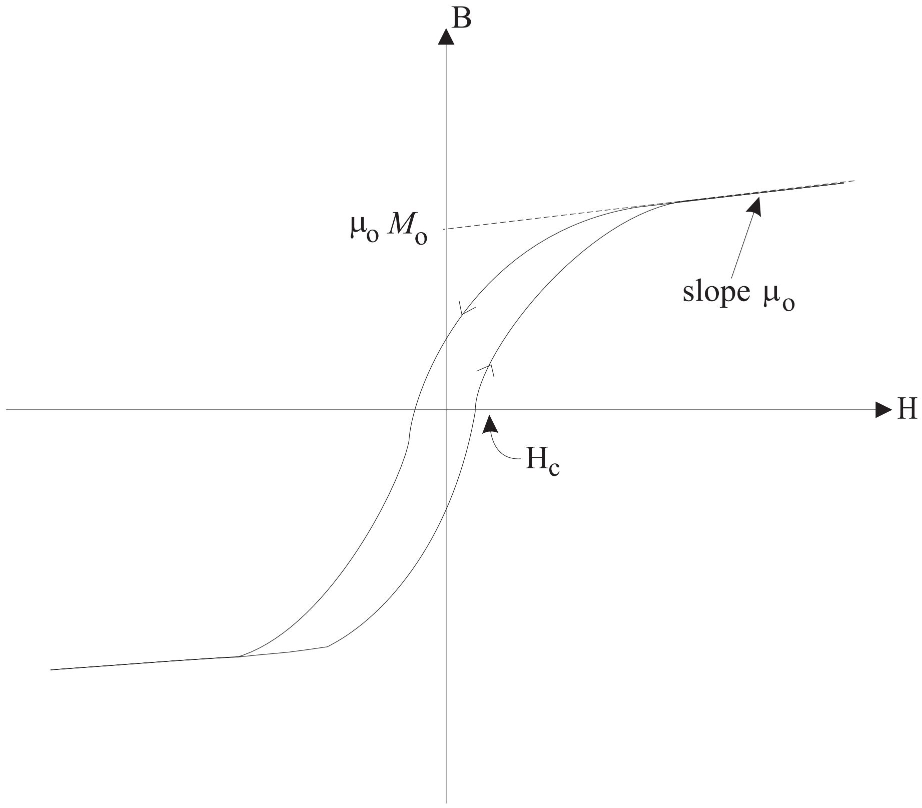


Figure 6.9: An hysteresis curve

6.7.4 Linear Lossy Ferromagnet

If we are prepared to consider small signals, and the sinusoidal steady state, we can model the behaviour of a material in which the magnetisation lags somewhat behind a sinusoidal magnetic field excitation by the equations

$$\mathbf{M} = (\chi_m^{'} - j\chi_m^{''})\,\mathbf{H} \tag{6.77}$$

and

$$\mathbf{B} = (\mu^{'} - j\mu^{''})\,\mathbf{H} \tag{6.78}$$

These equations are of the same form as used in the sinusoidal steady state in modelling linear lossy dielectric media. The linear lossy ferromagnet model is tantamount to replacing the hysteresis curve of Figure 6.9 by an ellipse. Although the model does not exhibit the saturation phenomenon shown in that figure, at least the energy loss property of the hysteresis curve is retained. The linear lossy ferromagnet model is widely used for the characterisation of ferrite media used in radio frequency communications systems.

6.7.5 Saturated Ferromagnet

We discuss first in this section some contrasts between the behaviour of dielectric and ferromagnetic media, and then expand upon the remarks made about ferromagnetic media in an earlier section.

In a dielectric medium the individual atoms may or may not have permanent dipole moments. When an electric field is supplied, atoms initially without a dipole moment may acquire one with a strength proportional to the electric field, and the polarisation described in Equation 6.62 results.

In the case of dielectric media in which the atoms already have in the absence of an internal electric field a dipole moment, thermal agitation normally ensures that these are randomly aligned, and in the absence of an electric field, no polarisation is macroscopically evident. When an electric field is applied a partial alignment, proportional to the strength of the internal field, of the dipole moments occurs, and again the macroscopically observed polarisation obeys the relation described in equation 6.62.

The situation in a ferromagnet is quite different. In a ferromagnetic medium, the individual atoms possess permanent magnetic moments, but as discussed earlier the phenomenon known as the exchange effect causes the dipole moments of several hundred to many thousand of adjacent atoms to remain exactly aligned. The result is that such groups of atoms are internally permanently and fully magnetised. In the absence of an applied field, there is no magnetisation evident on a macroscopic scale because the orientation of the magnetisation in different domains of the material is randomly distributed.

When magnetic fields are applied to the material, the domain wall boundaries move to cause the enlargement of domains with magnetisation direction similar to that of the applied field at the expense of domains in which the magnetisation is oppositely directed. How far the domain wall boundaries move is dependent upon the strength of the applied field, with the result that the magnetisation relations equation 6.72 and 6.76 are on a macroscopic scale observed. When however the applied field is made strong enough, the material becomes magnetised as a single domain in which the magnetisation has the same direction as the applied field and has a value equal to the magnetic moment of an individual atom multiplied by the number of atoms per unit volume. This terminal behaviour is described by the equations

$$|\boldsymbol{\mathcal{M}}| = M_0 \tag{6.79}$$

$$< \boldsymbol{\mathcal{M}} > \text{ is directed along } < \boldsymbol{\mathcal{H}}_0 > \tag{6.80}$$

where M_0 is called the *saturation magnetisation* of the material, $\boldsymbol{\mathcal{H}}_0$ is the externally applied field, and the $< >$ symbol represents the time average value. No further increase in the magnitude $|\boldsymbol{\mathcal{M}}|$ of the magnetisation is then possible. This behaviour has little resemblance to that of dielectric media.

What happens when time-varying magnetic fields of small amplitude are then superimposed upon the large steady magnetic field $\boldsymbol{\mathcal{H}}_0$ normally required for saturation bears even less resemblance to the corresponding behaviour in dielectric media. To understand what will happen, we must take note of a coupling which is known to exist at the atomic level between the *angular momentum* of the electrons in the material and the *magnetic moments* of those electrons. This coupling is responsible for the gyromagnetic behaviour discussed below.

If time-varying magnetic fields are applied in a saturated ferromagnet in a direction transverse to the magnetisation, they have the effect of producing on the magnetic moments of the electrons in the magnetised atoms a torque. This torque will produce a time rate of change of angular momentum of the electrons in a direction which is perpendicular both to the applied time-varying field and to the saturation magnetisation. The result is that the magnetisation will take up a precessional motion in which its magnitude does not change, its time average value is still along the large magnetising field $\boldsymbol{\mathcal{H}}_0$ producing the original saturation, but its instantaneous direction will be at some angle to that field.

Although a mathematical description of this behaviour is well beyond the scope of this course, its occurrence has been included as an illustration of the wide variety of material responses to the electromagnetic field which is present in nature; and so that the phenomenon, which will receive detailed description later in the course, will not then be too unexpected.

6.8 Depolarising and Demagnetising Factors

6.8.1 Field of a Polarised Sphere

Suppose a sphere is composed of a linear dielectric material and it is placed in a region of space in which there was previously, along the z axis, a uniform electric field $\boldsymbol{\mathcal{E}}_a$, called the *applied field* and caused by a distribution of sources *which does not change when the sphere is introduced.* We may ask what then is the electric field intensity inside and outside the sphere?

The field will consist of two parts:

- the applied field $\boldsymbol{\mathcal{E}}_a$, caused by the original sources which are assumed to be still in their original places; and

- an additional field $\boldsymbol{\mathcal{E}}_d$ caused by whatever sources are induced in the sphere.

The induced sources will consist of an induced surface charge density $\boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{n}}$ which is produced where the material comes to an abrupt end at its surface, and if the internal polarisation is non-uniform, will contain in addition an induced volume charge density $-\nabla \cdot \boldsymbol{\mathcal{P}}$.

We will now make a somewhat surprising assumption which will nevertheless be confirmed by the fact that it produces a self-consistent solution. We will suppose that in the interior of the sphere, despite the very curvilinear nature of the dipole field outside of the sphere, the polarisation is uniform if the originally applied field is uniform, and is along the z axis. Thus we assume that

$$\boldsymbol{\mathcal{P}}(x, y, z, t) = P_0\, \hat{\mathbf{u}}_z \tag{6.81}$$

It can be shown that this is correct solution, as the uniform polarisation will produce no induced volume charge density, and will produce a surface charge density at position $\mathbf{r}$ on the surface of

$$\rho_s^i = \boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{r}} = P_0 \cos\theta \tag{6.82}$$

If we solve for the field produced by this surface charge density inside the sphere, we find it is a spatially uniform field $\boldsymbol{\mathcal{E}}_d$ pointing in the opposite direction to that of the polarisation $\boldsymbol{\mathcal{P}}$, so that the total internal electric field

$$\boldsymbol{\mathcal{E}}_t = \boldsymbol{\mathcal{E}}_a + \boldsymbol{\mathcal{E}}_d \tag{6.83}$$

is also spatially uniform, and in a linear dielectric in which

$$\boldsymbol{\mathcal{P}} = \epsilon_0 \chi_e\, \boldsymbol{\mathcal{E}} \tag{6.84}$$

the polarisation is also spatially uniform, thus confirming assumption with which we started. The situation is therefore as pictured in Figure 6.10.

6.8.2 Depolarising and Demagnetising Factors

Pursuing in detail the analysis described in outline form above, it can be shown that for the polarised sphere the depolarising field $\boldsymbol{\mathcal{E}}_d$ is given by

$$\boldsymbol{\mathcal{E}}_d = -\frac{1}{3}\frac{\boldsymbol{\mathcal{P}}_0}{\epsilon_0} \tag{6.85}$$

We call the factor 1/3 the *depolarising factor* N_z for the z axis and write

$$\boldsymbol{\mathcal{E}}_d = -N_z \frac{\boldsymbol{\mathcal{P}}_0}{\epsilon_0} \tag{6.86}$$

A similar behaviour occurs for a sphere of uniformly magnetised material, which suffers a demagnetising field as a result of induced poles at the surface. The factor N_d is then called the *demagnetising factor*, and for a sphere has the same value of one third given above. The result we have just obtained can be generalised to shapes other than a sphere, but as the analysis is more difficult we will simply quote the results. These are that

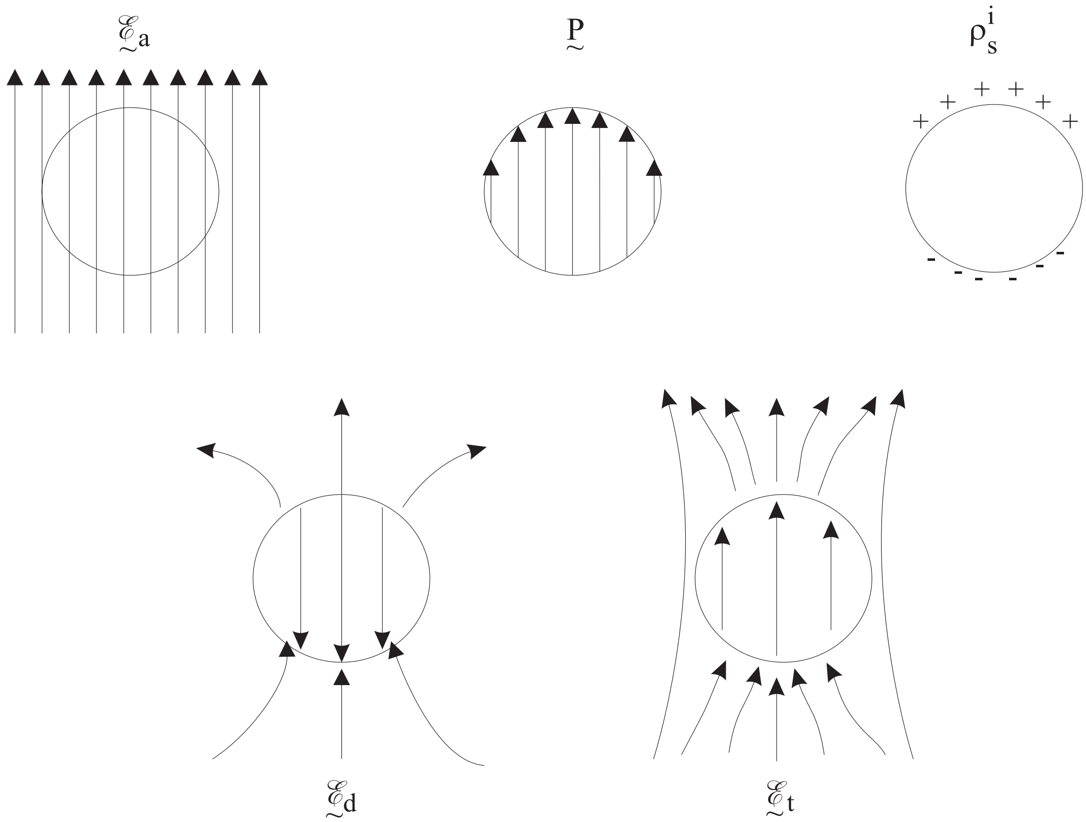


Figure 6.10: Various fields in a polarised sphere

- Any ellipsoidal linear dielectric or linearly permeable magnetic material placed in a uniform applied electric field $\boldsymbol{\mathcal{E}}_a$ or magnetic field $\boldsymbol{\mathcal{H}}_a$ as appropriate will become spatially uniformly polarised or magnetised.

- The abrupt change of polarisation or magnetisation at the surface produces there induced electric or magnetic charges.

- These induced electric or magnetic surface charges produce a contribution to the internal electric or magnetic field which is opposite in direction to the polarisation or magnetisation.

- If we introduce an xyz co-ordinate system along the principal axes of the ellipsoid, the components of the depolarising or demagnetising field can be calculated from the corresponding components of the polarisation or magnetisation by the equations

$$\begin{aligned} E_{dx} &= -N_x P_x/\epsilon_0 \\ E_{dy} &= -N_y P_y/\epsilon_0 \\ E_{dz} &= -N_z P_z/\epsilon_0 \end{aligned} \tag{6.87}$$

$$\begin{aligned} H_{dx} &= -N_x M_x \\ H_{dy} &= -N_y M_y \\ H_{dz} &= -N_z M_z \end{aligned} \tag{6.88}$$

  In the above equations we should note the appearance of the factor ϵ_0 in the first three, and the absence of a corresponding factor μ_0 in the second three.

- The depolarising and demagnetising factors are in general difficult to calculate but it can be shown that

$$N_x + N_y + N_z = 1 \tag{6.89}$$

This last equation makes the derivation of the depolarising or demagnetising factors for shapes possessing a high degree of symmetry a simple matter.

6.8.3 Demagnetising Factors of Simple Shapes

As an exercise, derive the depolarising or demagnetising factors for the simple shapes of:

- a sphere;
- a long thin rod whose axis is the z axis; and
- a thin disc normal to the z axis.

Chapter 7

BOUNDARY CONDITIONS

7.1 Introduction

Practical electromagnetic theory problems almost invariably involve finite geometries and discontinuities between the parameters characterising the medium in one section and those pertaining in another.

In a formal sense electromagnetic boundary conditions are required so that solutions to Maxwell's equations in differential form, which solutions involve the usual arbitrary constants, may be suitably matched as we cross such boundaries.

In a less formal sense, our knowledge of electromagnetic boundary conditions is required for another purpose. Any thorough understanding of electromagnetic theory must be based on a series of mental pictures of the possible electromagnetic field configurations which can occur in various geometries. Our knowledge of the requirements on electromagnetic field components at various plane boundaries, and in particular at metallic boundaries, is necessary for firstly the visualisation and secondly the validity checking of such potentially correct field pictures. It might be said that the source and vortex interpretation of Maxwell's equations in differential form, and a knowledge of the shortly to be derived results on boundary conditions, is with experience sufficient in most cases for the construction of a qualitatively correct field solution without detailed mathematical investigation.

7.2 Boundary Characterisation

We will for simplicity consider plane boundaries, and will regard any smoothly curved boundary as approximately plane at an appropriate scale of viewing. Such a plane boundary is shown in Figure 7.1 and is assumed to lie between region 1, in which the medium is characterized by real magnetic permeability μ_1, dielectric permittivity ϵ_1, and electric conductivity σ_1, and a region 2 in which the material is characterized by corresponding parameters μ_2, ϵ_2, and σ_2. A unit vector $\hat{\mathbf{n}}$ is directed normal to the boundary from region 1 to region 2. The reference directions for $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$ are similar to those for $\boldsymbol{\mathcal{H}}$ which are shown.

The boundary is assumed to carry a possible surface charge density ρ_s and a possible surface density $\boldsymbol{\mathcal{J}}_s$ which should be viewed as directed out of the paper.

In order to make a correct interpretation of some of the results to be derived below,

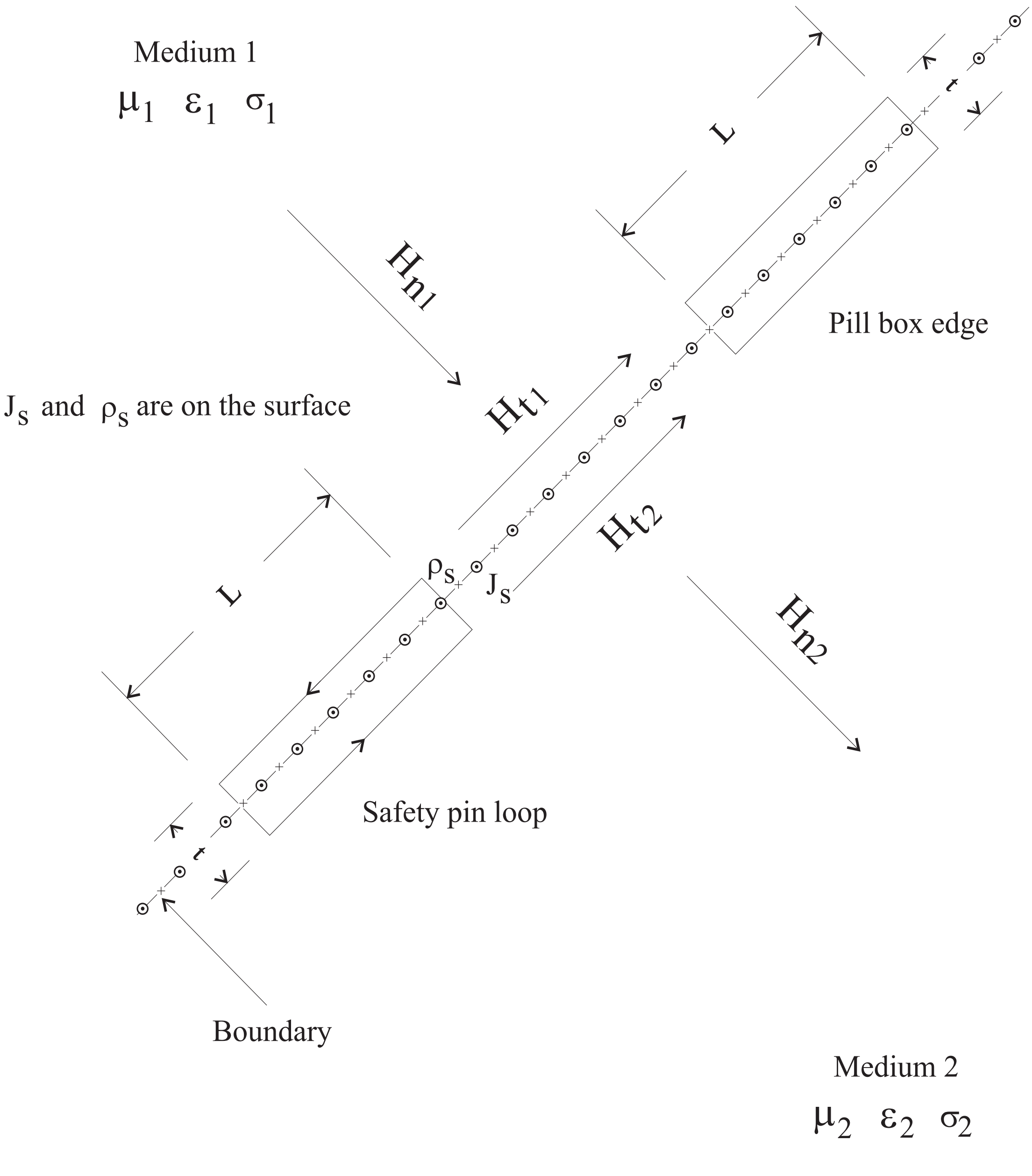


Figure 7.1: Variables and contours used in establishing electromagnetic boundary conditions

particular relationships between the quantities just defined and the reference directions for them as established in Figure 7.1 must be observed. It should be noted that the reference direction for surface current is out of the paper. The safety pin loop to which the line integrals will be applied lies with its long sides both in the plane of the boundary and in the plane of the paper, that is perpendicular to the direction of surface current. Moreover the direction of traverse of that contour, which is indicated by the arrows in Figure 7.1 and the reference direction for surface current, are related according to the right-hand rule. Finally, we note that the reference directions for tangential components on each side of the boundary will on one side match the direction of traverse of the contour but on the other side will be opposite. We will later take note of this fact in establishing signs of terms which appear in equations to be derived.

7.3 Maxwell's Equations Again

Because the results to be derived will come from Maxwell's equations, we take the opportunity to reproduce here, from Section **??**, Maxwell's equations for arbitrary media in both differential and integral form, both forms in the time domain. Firstly in differential form we have

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{7.1}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \tag{7.2}$$

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho \tag{7.3}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0 \tag{7.4}$$

while in integral form we have

$$\oint_C \boldsymbol{\mathcal{E}} \cdot \mathbf{dr} = -\int_S \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \cdot \mathbf{ds} \tag{7.5}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot \mathbf{dr} = \int_S \boldsymbol{\mathcal{J}} \cdot \mathbf{ds} + \int_S \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \cdot \mathbf{ds} \tag{7.6}$$

$$\oint_S \boldsymbol{\mathcal{D}} \cdot \mathbf{ds} = \int_v \rho \, dv \tag{7.7}$$

$$\oint_S \boldsymbol{\mathcal{B}} \cdot \mathbf{ds} = 0 \tag{7.8}$$

7.4 Method of Analysis

Because of the discontinuities in the material parameters, and the discontinuities in at least some components of the electromagnetic field which result therefrom, Maxwell's equations in differential form fail, in the sense that the derivatives do not exist, at the boundary. Maxwell's equations in integral form, however, still apply, as such finite discontinuities are readily integrable.

Maxwell's equations in integral form involve two contour integrals and two surface integrals. The method of analysis is to apply those equations to particular integrals over special contours or surfaces appropriately chosen in relation to the boundary. For the line integrals, the chosen contour is a *safety pin loop* of length L and thickness t of which one long side lies on each side of the boundary. This loop is shown in Figure 7.1.

In the case of the surface integrals, the chosen surface is the *pill box surface.* In the pill box, the large flat surfaces are of dimensions $L \times L$ and lie parallel to and on each side of the boundary. The box has thickness t. In Figure 7.1 the large $L \times L$ surfaces are viewed from an edge and appear as a line.

7.5 The General Case

It is a simple matter, and should be taken as an exercise, to show that Maxwell's equations in integral form applied to these contours lead to the results:-

$$\boldsymbol{\mathcal{E}}_{t2} - \boldsymbol{\mathcal{E}}_{t1} = 0 \tag{7.9}$$

$$D_{n2} - D_{n1} = \rho_s \tag{7.10}$$

$$\hat{\mathbf{n}} \times (\boldsymbol{\mathcal{H}}_{t2} - \boldsymbol{\mathcal{H}}_{t1}) = \boldsymbol{\mathcal{J}}_s \tag{7.11}$$

$$B_{n2} - B_{n1} = 0 \tag{7.12}$$

In the above equations the subscript t indicates a tangential component and subscript n a normal component of the relevant field.

In the two centre equations, note must be taken of the order of the terms on the left hand side. In Equation 7.10, the term which appears with the positive sign is that expressing the outward component of the normal vector from the medium in question, while the term appearing with the negative sign expresses an inward vector to the medium in question. In Equation 7.11 the term which appears with the positive sign is that in which reference direction established for tangential field and the sense of the contour match, while the term for which the negative sign appears has its reference direction opposite to the sense with which the contour is to be traversed.

The results which have just been derived are the most general expression of electromagnetic boundary conditions and as such are always valid. In words they state that the tangential component of electric field intensity and the normal component of magnetic flux density are always continuous across a boundary, while the normal component of electric flux density and the tangential component of magnetic field intensity can suffer discontinuities if surface charges or currents are present.

Whether such surface charges or currents can in fact be present is determined by particular properties of the media present on each side of the boundary, and will be discussed in particular cases below.

The general case which we have just discussed can be particularised in two directions. Firstly the quite arbitrary time dependence assumed so far could be replaced by either no time dependence ie the electrostatic or magnetostatic situation, or by a sinusoidal steady state time dependence. In another aspect, the general case may be particularised by assuming particular values for the material parameters. For example materials may be idealised as perfect insulators, perfect conductors, or simply media with finite (including zero) conductivity. These particularisations can create a potentially large number of cases through which we will try to pick our way with care.

It will be convenient firstly to consider particularisations of the media, and with each of those cases then to consider particularisations of the time dependence.

7.6 Imperfect Conductors

7.6.1 Definition

We regard an imperfect conductor as a medium with a finite (including zero) conductivity. Insulators will under this definition be a sub-class of imperfect conductors.

When both media are imperfect conductors, we consider in the next two sections the possibilities of having a surface current or a surface charge density, and then examine in the following section the impact upon the boundary conditions.

7.6.2 Surface Currents

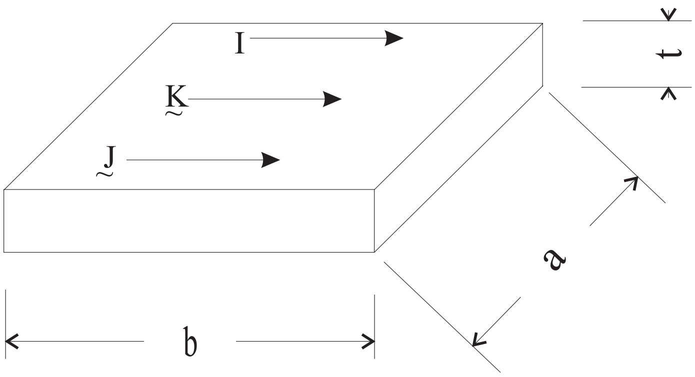


Figure 7.2: Conducting slab

A surface current can be regarded as the limiting case when a finite amount of current I flows in a thin slab of area dimensions a and b thickness t, and we let $t \to \infty$.

If while t is small but not zero we describe this situation in terms of a volume current density $\mathcal{J}$ or a surface current density $\mathcal{K}$. We may relate these quantities to I by

$$I = Ka = Jat \tag{7.13}$$

If the material has electric resistivity ρ (note this symbol does not for the moment represent volume charge density) the resistance of the slab is

$$R = \frac{\rho\, b}{at} \tag{7.14}$$

The power $P = I^2 R$ dissipated in the slab is therefore given by

$$P = \frac{K^2 a^2 b \rho}{at} \tag{7.15}$$

$$P = \frac{K^2 a b \rho}{t} \tag{7.16}$$

If K and ρ are non-zero, ie we have a surface current and the conductor is not perfect, then $P \to \infty$ as $t \to 0$. Since we cannot produce an infinite amount of power we must have $K = 0$, ie we cannot have a surface current density in an imperfect conductor.

7.6.3 Consequences at Boundary

The absence of a surface current thus reduces the boundary conditions to

$$\boldsymbol{\mathcal{E}}_{t2} - \boldsymbol{\mathcal{E}}_{t1} = 0 \tag{7.17}$$

$$D_{n2} - D_{n1} = \rho_s \tag{7.18}$$

$$\hat{\mathbf{n}} \times (\boldsymbol{\mathcal{H}}_{t2} - \boldsymbol{\mathcal{H}}_{t1}) = 0 \tag{7.19}$$

$$B_{n2} - B_{n1} = 0 \tag{7.20}$$

These are not much changed from equations 7.9 to 7.12.

7.6.4 Possibility of Surface Charge

We note in particular that a surface charge density ρ_s can exist, in many situations, even though not all.

It can for example exist as a dc value on the surface of any material, ie perfect insulator, (after it has been rubbed with the cat), on a non-insulating imperfect conductor, or on a perfect conductor.

It can also exist as an ac value on the surface of a perfect conductor or an imperfect conductor, as even when surface currents are outlawed in the latter, volume currents directed perpendicular to the surface can make a surface change density change.

The case when ρ_s is outlawed is when the fields and other variables are all sinusoidal and both materials are perfect insulators. There can in this situation be no mechanism of charge transport to change the surface charge density.

7.7 Two Insulating Media

The discussion above has made it clear that we cannot in this case have a surface current density $\boldsymbol{\mathcal{J}}_s$ and neither can we have a time-varying surface charge density ρ_s. We can have an unvarying ρ_s.

7.8 One Perfect Conductor

7.8.1 Perfect Conductor Concept

We take a practical view of the term *perfect conductor* to mean a material with very high conductivity, such as a metal, within which any electric field must be negligibly small, but we can still establish a steady value of magnetic field $\boldsymbol{\mathcal{H}}$ and magnetic flux density $\boldsymbol{\mathcal{B}}$, if we are prepared to spend a reasonable time doing it.

We are thus not discussing super-conducting media from which magnetic field are expelled through processes different from those considered in the course so far.

7.8.2 Possible Interior Fields

We are going to make medium 1 the perfect conductor. In this medium there can be no electric field either time varying or static and thus no electric flux density either time varying or static.

Because **curl** $\boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}$ there can be no time varying magnetic flux density $\boldsymbol{\mathcal{B}}$ and hence in this assumed linear medium, no time varying magnetic field $\boldsymbol{\mathcal{H}}$. There can however be a steady magnetic flux density $\boldsymbol{\mathcal{B}}$ and magnetic field $\boldsymbol{\mathcal{H}}$. In practice these fields take a long time to establish or to change to a new steady value, the time depending on just how large the conductivity of the material is.

7.8.3 Consequences at Boundary

As said above we make medium 1 the perfect conductor. This is so that the vector $\hat{\mathbf{n}}$ points out into the adjacent space. For dc fields we would have the forms already quoted as equation 7.9 to 7.12 with $\boldsymbol{\mathcal{E}}_{t1}$ and D_{n1} set to zero, viz

$$\boldsymbol{\mathcal{E}}_{t2} = 0 \tag{7.21}$$

$$D_{n2} = \rho_s \tag{7.22}$$

$$\hat{\mathbf{n}} \times (\boldsymbol{\mathcal{H}}_{t2} - \boldsymbol{\mathcal{H}}_{t1}) = \boldsymbol{\mathcal{J}}_s \tag{7.23}$$

$$B_{n2} - B_{n1} = 0 \tag{7.24}$$

For time varying fields we will consider in particular the sinusoidal steady state in which we represent by black capitals as shown below.

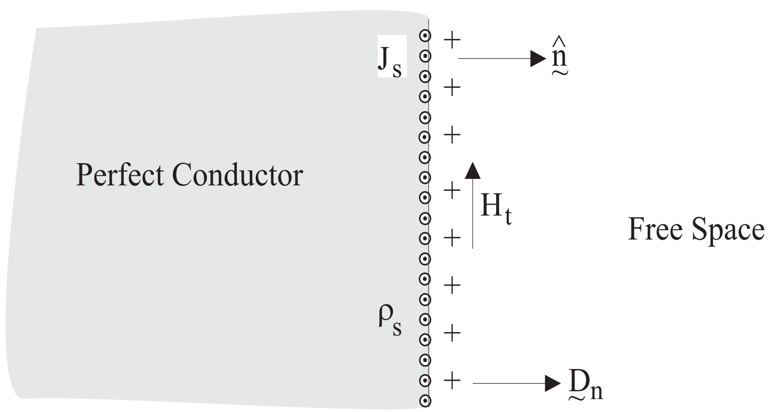


Figure 7.3: Boundary conditions at a perfect conductor surface

Because in the sinusoidal steady state all fields in the perfect conductor are now zero we will drop the subscript 2 from the fields in the adjacent space.

In the light of the above discussion and realising that both $\mathbf{J}_s$ and ρ_s are allowed we now have boundary conditions in the form

$$\mathbf{E}_t = 0 \tag{7.25}$$

$$\mathrm{D}_\mathrm{n} = \rho_s \tag{7.26}$$

$$\hat{\mathbf{n}} \times \mathbf{H}_t = \mathbf{J}_s \tag{7.27}$$

$$\mathrm{B}_\mathrm{n} = 0 \tag{7.28}$$

This final set of boundary conditions is the most commonly encountered in electromagnetic theory.

We note in relation to Figure 7.3 above the reference directions for $\boldsymbol{\mathcal{J}}_s$ and $\boldsymbol{\mathcal{H}}_t$ are easily established through the right hand rule applied to $\boldsymbol{\mathcal{J}}_s$.

Chapter 8

ENERGY AND POWER

8.1 Introduction

In this chapter we will consider the storage and transformation of energy in electromagnetic systems.

In the evolution of physical theory, two of the most durable principles have been that of *conservation of charge*, (which we in fact took in Chapter 3 as the very starting point of our conceptual scheme for electromagnetic theory), and that of *conservation of energy.* Although the latter has in this century been generalised to the concept of conservation of mass-energy, we are not in this course concerned with mass-energy transformations, and our view of the principle will be simply that of conservation of energy.

The principle of conservation of energy, and the fact that electromagnetic fields can exert forces which can do work, inevitably leads to the idea that energy can be stored in electromagnetic systems. Pursuing mathematical formulae for the amount of energy stored leads to the discovery that very general expressions for that quantity can be developed in terms of the field variables, rather than in terms of the distribution of sources. A simple way of looking at this result is that the energy is stored *within the field itself.* This viewpoint is very commonly adopted in the study of energy and power flow in electromagnetic systems.

8.1.1 Level of Treatment

In this treatment of energy storage and power flow, some results will be advanced by inductive generalisation of formulae which can be estabilshed readily only in simple geometries. In this aspect the Chapter may appear to differ from earlier chapters, where an attempt was made to produce results by deduction from basic principles. Such a difference, if it does exist, should not be seen as undermining the validity of the results to be presented here. In any scientific work, the acceptance of any theory is only justified by its continuing to correctly predict experimentally verifiable results. The validity of the general formulae to be offerred below is amply justified by their success in surviving empirical test.

8.2 Methods of Energy Input

In a general electromagnetic system we can identify the following methods of exchanging energy between that system and its environment.

- Moving charged bodies in an electric field. The forces on the charged bodies result in the performance of mechanical work.
- Injecting currents from voltage sources into terminals of a network. It is usual to regard this as an example of the delivery of electrical power.
- Moving polarised dielectrics in electric fields. The forces and torques on the polarised bodies result in the performance of mechanical work.
- Moving current carrying wires in a magnetic flux density. The forces on the wires require the performance of mechanical work. *It is also true that the movement of the wires may cause changing magnetic fields which induce voltages in the circuits in which the currents flow, causing an exchange of electric energy of the type described above.*
- Moving magnetised media in a magnetic field. The forces and torques exerted by the field on the poles of the media involve the performance of mechanical work. It is also true that changing fluxes caused by the movement may induce voltages in circuits which may be carrying currents to produce the magnetic field in which the motion is taking place.

Faced with the complexities involved in the above description it would appear to be wise to consider first some simple cases.

8.3 Simple Energy Storage Formulae

8.3.1 Linear Electrostatic Case

In the simple case of a distribution of point charges which are assembled from infinitely separated points into a final position in a linear homogeneous dielectric medium of dielectric permittivity ϵ, it is possible to show from a study of the mechanical work done that the total stored energy, which we define as equal to the mechanical work done, is given by

$$U_e = \frac{1}{2}\,\epsilon \int_v \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\, dv \tag{8.1}$$

where the integral is over the entire region of the field. One of several possible alternative versions of this formula is clearly

$$U_e = \frac{1}{2} \int_v \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{D}}\, dv \tag{8.2}$$

In another simple case when an initially uncharged parallel plate capacitor is charged from a suitably variable voltage source the total stored energy, which we reckon is equal to the electrical energy delivered by the source, is given by the same two formulae quoted above.

In this second example we may see a basis for regarding the second form of the above two formulae as being more closely related to the physical processes. This is because during the charging process the incremental component of the work done is obviously given by vdq where v is the voltage so far developed across the capacitor and dq is the element of charge being added. In this expression there is a direct relation between v and $\boldsymbol{\mathcal{E}}$, the electric field within the capacitor, and there is also a direct relation between dq and $\mathbf{d}\boldsymbol{\mathcal{D}}$, the change in electric flux density in the capacitor. Thus as far as the incremental component of work done in this change it seems that the most directly appropriate formula is

$$dU_e = \int_v \boldsymbol{\mathcal{E}} \cdot \mathbf{d}\boldsymbol{\mathcal{D}}\, dv \tag{8.3}$$

rather than any alternative formula in which we might substitute for $\boldsymbol{\mathcal{D}}$ in terms of $\epsilon\boldsymbol{\mathcal{E}}$ or for $\boldsymbol{\mathcal{E}}$ in terms of $\boldsymbol{\mathcal{D}}/\epsilon$. This argument, which might appear a little thin in the linear dielectric case, becomes quite compelling in the non-linear dielectric case where such substitutions are not possible.

8.3.2 Linear Magnetostatic Case

A not entirely parallel examination of the work done, perhaps by mechanical or perhaps by electrical means, to create a magnetic field leads to the result

$$U_m = \frac{1}{2}\,\mu \int_v \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{H}}\, dv \tag{8.4}$$

Of course in the case of a linear medium this result has several alternative forms, one of which is

$$U_m = \frac{1}{2} \int_v \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{B}}\, dv \tag{8.5}$$

To see which of these results appears to be the closer to physical processes, we consider the case where the field results from the creation in an inductor of a current by means of a suitably variable voltage generator connected to the inductor terminals. In any portion of the field creation process, we find the power being delivered to the circuit is the product of the total current i (which is related to the total magnetic field $\boldsymbol{\mathcal{H}}$) and the currently induced voltage v (which is related to the time rate of change of the flux density $\boldsymbol{\mathcal{B}}$).

Thus in an incremental change of flux taking place over time δt, we will find the component $\delta U_m = vi\delta t$ of energy change is related to the product of $\boldsymbol{\mathcal{H}}$, $\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}$ and δt ie to the product of $\boldsymbol{\mathcal{H}}$ and $\delta\ \boldsymbol{\mathcal{B}}$. Thus as far as the incremental component of energy supplied to bring about this field change is concerned, it seems that the most directly appropriate formula is

$$dU_m = \int_v \boldsymbol{\mathcal{H}} \cdot \mathbf{d}\boldsymbol{\mathcal{B}}\, dv \tag{8.6}$$

8.4 General Formula for Energy Change

In the light of the results discussed in the last two sections we will postulate the general expression for the change in stored energy in an electromagnetic system in response to changes in fields is

$$dU = \int_v \{\boldsymbol{\mathcal{E}} \cdot \mathbf{d}\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{H}} \cdot \mathbf{d}\boldsymbol{\mathcal{B}}\} dv \tag{8.7}$$

This general result is, in simple situations amenable to analysis, in accord with theoretical results based on Maxwell's equations and the force law, and in more complex geometries is in accord with experiments, and will thus be regarded as a correct statement in all cases of the changes in stored energy in a field.

Because it is very convenient to regard the energy as actually stored within the field, we will remove the integration and say that at any point the change in energy stored *per unit volume* in a field, when the field changes is

$$dW = \boldsymbol{\mathcal{E}} \cdot \mathbf{d}\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{H}} \cdot \mathbf{d}\boldsymbol{\mathcal{B}} \tag{8.8}$$

where W denotes the energy stored per unit volume of space at a point at which the field vectors appearing in the equation above apply.

If we divide by an incremental time dt over which the energy charge has taken place and proceed to the limit, recognising the independence of both the space and time variables, we obtain

$$\frac{\partial W}{\partial t} = \boldsymbol{\mathcal{E}} \cdot \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{H}} \cdot \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{8.9}$$

8.5 Energy Change Integral

8.5.1 Analysis

Equations of the general form appearing above, ie where the right hand side contains products of the field vectors and their space or time derivatives, occur in all levels of the electromagnetic theory course. The equation above is probably the first example of this recurring phenomenon. In all such cases we find that Maxwell's equations in differential form, which contain on the one side spatial derivatives and on the other side time derivatives, allow a transformation of the equation to another form, so that progress toward establishing a theorem is made. In this case we use Maxwell's equations to substitute for the time derivatives of the fields and obtain

$$\frac{\partial W}{\partial t} = \boldsymbol{\mathcal{E}} \cdot (\mathbf{curl}\, \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{J}}) - \boldsymbol{\mathcal{H}} \cdot \mathbf{curl}\, \boldsymbol{\mathcal{E}} \tag{8.10}$$

Moving the term involving the volume current density $\boldsymbol{\mathcal{J}}$ to the left hand side we can recognise the terms which remain on the right hand side as being (with a sign change) the components of the expansion of a vector product. Thus

$$\frac{\partial W}{\partial t} + \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}} + \mathrm{div}\, (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) = 0 \tag{8.11}$$

8.5.2 Interpretation

To form an interpretation of this last result, we examine the terms in order, and apply the conservation of energy concept discussed at the beginning of the chapter.

The first term in the above equation is, from the material presented so far, the time rate of change in the stored energy per unit volume, at or in the vicinity of the point for which the field vectors are defined. The second term may be recognised, at least in the case when the currents are conduction currents, as the rate at which energy is dissipated per unit volume per unit time in the medium as a result of its resistance. (If this result is unfamiliar, try drawing a small rectangular box with one of its edges in the direction of the electric field, and calculate the product of the voltage V between opposite faces and the current I passing through them, and divide by the volume of the box). If we then apply the conservation of energy concept, we are obliged to interpret the third term as *the rate at which energy per unit volume is leaving the region under study.*

The equation above is in point form. In volume integral form it is

$$\int_v \frac{\partial W}{\partial t}\,dv + \int_v \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}\,dv + \int_v \mathrm{div}\,(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}})\,dv = 0 \tag{8.12}$$

We now apply Gauss' theorem to the last term, to obtain

$$\oint_S (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \cdot \mathbf{ds} = \int_v \mathrm{div}\,(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}})\,dv \tag{8.13}$$

In view of the interpretation we have for the right hand side, we must now interpret the left hand side as the rate at which energy is leaving the volume v through the surface S via electromagnetic means. If we wish to pursue the interpretation further, we might note that the integral over the entire surface is merely the sum of contributions from separate small elements of that surface and boldly apply *to each element of the surface* the same interpretation as we have applied to the surface as a whole. Thus we say that the kernel $\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}$ of the integral above is the rate (per unit time) at which energy is leaving the surface *per unit area of that surface.*

8.6 Real Poynting vector

8.6.1 Definition

In the light of the interpretation above we define the *Poynting vector*

$$\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} \tag{8.14}$$

8.6.2 Interpretation

The Poynting vector has units of Wm^{-2}, and as shown above is taken to give the magnitude and direction of power flow per unit area carried by an electromagnetic field.

8.6.3 Validation

In the discussion leading to the derivation of the Poynting vector, the reasoning contains several inductive steps, particularly in the transfer of, an interpretation of, integrals initially over complete volumes or closed surfaces to each small element of those volumes or surfaces. These steps are acceptable in the search for plausible results, but are only fully justified by their continued success in predicting the results of experiment.

The correctness of the interpretation of the Poynting vector as giving an indication of power flow per unit area is amply justified in the context of radio communication. As we shall see in the next chapter, Maxwell's equations predict the existence of electromagnetic waves which we know are widely used for communication. An electromagnetic wave has associated with it both an electric field $\boldsymbol{\mathcal{E}}$ and a magnetic field $\boldsymbol{\mathcal{H}}$ and from these we can construct a Poynting vector. It is easy to show both theoretically and experimentally that the wave propagates in the direction in which the Poynting vector *points*, and that in order to receive a strong signal we need to place a receiving antenna in a place where the density of power flow as expressed by the Poynting vector is strongest.

8.7 Complex Poynting Vector

In the discussion of this chapter so far the variables have all been real quantities with quite arbitrary (or perhaps no) time variation; complex phasors have not so far (in this chapter) raised their pretty heads. Because of the usefulness of sinusoidal steady state analysis and the convenience of employing phasor representation in that analysis, they will do so now.

We must realise that in the previous analysis when $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are time-varying quantities, so will be the energy density per unit volume W and the Poynting vector $\boldsymbol{\mathcal{N}}$.

In sinusoidal steady state analysis if we define a Poynting vector in terms of phasor variables, which as we know do not have a time variation, then the newly defined Poynting vector will not have a time variation either. But it will have a very useful relationship with the originally defined real time-varying Poynting vector. This relationship will be derived after the new vector is defined.

8.7.1 Definition

We define the *Complex Poynting Vector* as

$$\mathbf{S} = \frac{1}{2}\,\mathbf{E} \times \mathbf{H}^* \tag{8.15}$$

8.7.2 Interpretation

In this equation the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are now the complex phasors $\mathbf{E}$ and $\mathbf{H}$ representing the real fields. Note the appearance of the complex conjugate symbol in the equation. In the analysis below it should be clear which symbols represent the real fields and which represent complex quantities. Where space permits the display of functional dependencies is shown. Note that the real time-varying quantities are on the left of the equality sign and the complex quantities on the right.

It must be realised that although the complex Poynting vector is a complex number and is composed of phasors it is not itself a phasor; we shall see that it does not represent any quantity which has a sinusoidal variation at the angular frequency ω of the fields. With this understanding of the meaning of symbols, we may note that since

$$\boldsymbol{\mathcal{E}}(x, y, z, t) = \frac{\mathbf{E}(x, y, z)e^{j\omega t} + \mathbf{E}^*(x, y, z)e^{-j\omega t}}{2} \tag{8.16}$$

and

$$\boldsymbol{\mathcal{H}}(x, y, z, t) = \frac{\mathbf{H}(x, y, z)e^{j\omega t} + \mathbf{H}^*(x, y, z)e^{-j\omega t}}{2} \tag{8.17}$$

$$\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} = \frac{\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}}{4} + \frac{\mathbf{E} \times \mathbf{H}\, e^{2j\omega t} + \mathbf{E}^* \times \mathbf{H}^*\, e^{-2j\omega t}}{4} \tag{8.18}$$

If we take the time average of each side, the second term on the right hand side contributes zero and hence

$$< \boldsymbol{\mathcal{N}} >= \frac{\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}}{4} \tag{8.19}$$

ie

$$< \boldsymbol{\mathcal{N}} >= \frac{\mathbf{S} + \mathbf{S}^*}{2} \tag{8.20}$$

ie

$$< \boldsymbol{\mathcal{N}} >= \Re\{\mathbf{S}\} \tag{8.21}$$

The definition in equation 8.15 was chosen so that this relation would result.

Although in transfer of power by electromagnetic means the power flow per unit area or energy stored per unit volume are factors of time, it is generally the average value over time in which we are interested: when we cook our dinner it is the time average power we pay for, and in electromagnetic communication it is the time average power which sets the signal to noise ratio and thus the error rate.

The practical utility of the complex Poynting vector is that it provides an economical means of calculating, for sinusoidal steady state fields, the time average of the power flow, without our having to become involved in the details of the time variation.

Chapter 9

ELECTROMAGNETIC WAVES

9.1 Introduction

In this section we will show that the equations of electrodynamics lead in a natural way to the prediction of electromagnetic waves.

In showing this we will make use of the simplest wave solution, that of the uniform plane wave.

For this solution we will derive some fundamental results concerned with the relative amplitudes and spatial organisation of its fields, the relation of wave length to frequency, and the power carried by the wave.

It will be shown that many of these properties are preserved when more complicated solutions are generated by forming a superposition of various plane wave solutions.

9.2 Fundamental Equations

9.2.1 Maxwell's Equations

We will suppose the medium we are studying is either free space or a *homogeneous lossless linear medium*, ie it is characterized by a constant μ and ϵ, and that μ and ϵ are real and σ is zero. We assume also that there are no free charges or currents. Then Maxwell's equations in the frequency domain become:

$$\mathbf{curl}\,\mathbf{E} = -j\omega\mu\,\mathbf{H} \tag{9.1}$$

$$\mathbf{curl}\,\mathbf{H} = j\omega\epsilon\,\mathbf{E} \tag{9.2}$$

$$\mathrm{div}\,\epsilon\mathbf{E} = 0 \tag{9.3}$$

$$\mathrm{div}\,\mu\mathbf{H} = 0 \tag{9.4}$$

9.2.2 Relevance

In this instance the divergence equations are already implied by the **curl** equations, and we will therefore not consider them further.

9.2.3 Helmholz Equation

If we combine the two curl equations we find

$$\mathbf{curl}\,\mathbf{curl}\mathbf{E} = \omega^2 \epsilon\mu \mathbf{E} \tag{9.5}$$

The vector identity **curl curl** = **grad** *div* − ∇^2 gives, in view of the fact that *div* $\mathbf{E} = 0$,

$$\nabla^2 \mathbf{E} = -\omega^2 \epsilon\mu \mathbf{E} \tag{9.6}$$

This equation is known as the *three-dimensional wave equation* and we will be concerned with various of its solutions. When any solution for **E** is known we can calculate the accompanying **H** from equation 9.1 above.

9.3 Wave Terminology

9.3.1 Exponential Solutions

This being a second order homogeneous differential equation, we will look for waves with an exponential spatial variation. That is for solutions of the type

$$\mathbf{E} = \mathbf{E}_0 \, e^{-\vec{\gamma}\cdot\mathbf{r}} \tag{9.7}$$

where $\mathbf{E}_0$ is a complex phasor independent of position, giving the value of the electric field phasor at the origin.

9.3.2 Propagation Vector

In the above equation $\vec{\gamma}$ is a complex vector which we call the *propagation vector.* It can be decomposed into real and imaginary component vectors as:-

$$\begin{aligned} \vec{\gamma} &= \vec{\alpha} + j\vec{\beta} && (9.8)\\ \vec{\alpha} &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} && (9.9)\\ \vec{\beta} &= \beta_x \mathbf{i} + \beta_y \mathbf{j} + \beta_z \mathbf{k} && (9.10) \end{aligned}$$

So we see the spatial variation of **E** is as the product of factors:

$$e^{-\vec{\alpha}\cdot\mathbf{r}} \, e^{-j\vec{\beta}\cdot\mathbf{r}} \tag{9.11}$$

9.3.3 Plane Wave Terminology

In the above equation the first factor changes amplitude, the second changes phase. The directions of maximum rates of change of amplitude and phase are alpha and beta respectively. The plane perpendicular beta is a *plane of constant phase*; that such planes exist is why solutions of this type are called *plane waves.* The plane perpendicular alpha is a *plane of constant amplitude.* If such variations of amplitude exist, ie if α not equal to 0, the wave is called a *non-uniform plane wave.* If no such variation of amplitude exists, ie if $\alpha = 0$, the wave is called a *uniform plane wave.*

9.4 Uniform Plane Wave Solutions

9.4.1 Simplification of Maxwell's Equations

When $\vec{\alpha} = \vec{0}$ and the z axis is chosen to lie along the direction of $\vec{\beta}$, we can assert the equations: $\frac{\partial}{\partial x} = 0$, $\frac{\partial}{\partial y} = 0$, and $\frac{\partial}{\partial z} = -j\beta$. With these restrictions Maxwell's equations in the frequency domain become:

$$\begin{aligned} j\beta \mathrm{E_y} &= -\mu j\omega \mathrm{H_x} &(9.12)\\ -j\beta \mathrm{E_x} &= -\mu j\omega \mathrm{H_y} &(9.13)\\ 0 &= -\mu j\omega \mathrm{H_z} &(9.14)\end{aligned}$$

$$-j\beta \mathrm{E_z} = 0 \qquad (9.15)$$

$$\begin{aligned} j\beta \mathrm{H_y} &= \epsilon j\omega \mathrm{E_x} &(9.16)\\ -j\beta \mathrm{H_x} &= \epsilon j\omega \mathrm{E_y} &(9.17)\\ 0 &= \epsilon j\omega \mathrm{E_z} &(9.18)\end{aligned}$$

$$-j\beta \mathrm{H_z} = 0 \qquad (9.19)$$

9.4.2 Transverse Electromagnetic Wave Solutions

We have above eight equations; four of them require and are satisfied by setting both longitudinal components to zero, ie $\mathrm{E_z} = 0$ and $\mathrm{H_z} = 0$. Because of these conditions the resulting waves are called *TEM* waves. The remaining four equations, when the j factors are dropped, can be grouped as the two pairs shown below.

$$\begin{aligned} \beta \mathrm{E_y} &= -\mu\omega \mathrm{H_x} &(9.20)\\ \beta \mathrm{H_x} &= -\epsilon\omega \mathrm{E_y} &(9.21)\end{aligned}$$

and

$$\begin{aligned} \beta \mathrm{E_x} &= \mu\omega \mathrm{H_y} &(9.22)\\ \beta \mathrm{H_y} &= \epsilon\omega \mathrm{E_x} &(9.23)\end{aligned}$$

The pairs of equations above represent independent solutions with spatial arrangements of **E**, **H** and $\vec{\beta}$ as illustrated in Figure 9.1.

We have one solution as illustrated for each pair of equations. In each case the equations when combined lead to the result:-

$$\beta^2 = \mu\epsilon\omega^2 \qquad (9.24)$$

Figure 9.1: Mutually orthogonal **E**, **H** and $\vec{\beta}$

The wave velocity $c = \omega/\beta$ is then given by

$$c = \frac{1}{\sqrt{\mu\epsilon}} \tag{9.25}$$

9.4.3 Detailed Expression of Solutions

In terms of the physically meaningful time dependent variables the detailed solutions are therefore:

$$\mathrm{E_y} = \Re\left\{\mathrm{E_0^{(1)}e^{j(\omega t-\beta z)}}\right\} \tag{9.26}$$
$$\mathrm{H_x} = \Re\left\{-\mathrm{H_0^{(1)}e^{j(\omega t-\beta z)}}\right\} \tag{9.27}$$

and

$$\mathrm{E_x} = \Re\left\{\mathrm{E_0^{(2)}e^{j(\omega t-\beta z)}}\right\} \tag{9.28}$$
$$\mathrm{H_y} = \Re\left\{\mathrm{H_0^{(2)}e^{j(\omega t-\beta z)}}\right\} \tag{9.29}$$

where the amplitudes $\mathrm{E_0^{(i)}}$ and $\mathrm{H_0^{(i)}}$ of the wave are related by $\mathrm{E_0/H_0} = \mu\omega/\beta = \beta/(\epsilon\omega) = \sqrt{(\mu/\epsilon)}$

9.4.4 Characteristic Impedance of Medium

The ratio of the electric field and the magnetic field phasors calculated above is called the *wave impedance* of the medium and we denote it by η. Thus

$$\frac{\mathrm{E_0}}{\mathrm{H_0}} = \eta \tag{9.30}$$

where

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{9.31}$$

9.4.5 Remarks on Polarization

The polarisation of the wave is by convention defined by the motion of **E**. Both of the two independent solutions derived above are examples of *linear polarization*, one along the x axis and one along the y axis. In each case **E**, **H**, and $\vec{\beta}$ form a right hand set of mutually orthogonal vectors.

The general solution for a uniform plane wave can be obtained by superposing any combination, with arbitrary relative phases and amplitudes, of these two solutions. The result is that **E** then describes an elliptical path with arbitrary orientation and eccentricity. Special cases include those of *linear* or *circular* polarization. In all cases the **H** vector describes a path of the same shape, rotated 90 degrees about the z axis in the right hand sense. The ratio of the peak values of the electric field and magnetic field is a constant equal to the characteristic impedance η defined above.

We note that either sense (ie positive or negative) of circular polarization can be synthesised by this procedure.

9.5 Power Flow in Uniform Plane Waves

9.5.1 Calculation

For the uniform plane waves in the forward direction the complex Poynting vector of Section reference has the value:

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ \mathrm{E_x H_y^*} - \mathrm{E_y H_x^*} \end{bmatrix} \tag{9.32}$$

.

Since$\mathrm{H_y} = \mathrm{E_x}/\eta$ we may simplify this to

$$\mathbf{S} = \frac{1}{2\eta}\begin{bmatrix} 0 \\ 0 \\ \mathrm{E_x E_x^*} + \mathrm{E_y E_y^*} \end{bmatrix} \tag{9.33}$$

.

It is easy to see that the real part of **S** is the same vector.

9.5.2 Interpretation

The form of the above expression shows that the power flow is positive in the $+z$ direction, and that the two components of the electric field do not interact.

A similar analysis of power flow in the case of two waves propagating in opposite directions leads to the similar and interesting conclusion that these two waves do not interact in their power flow either. The particular result we can prove is that the nett power carried to the right is equal to the power which the rightward propagating wave (if it alone were present) would carry, less the power which the left-propagating wave would carry to the left if it alone were present.

This result demonstrates a property know as the *power orthogonality of oppositely directed uniform plane waves.*

Because power is a quantity *quadratically* dependent on the field amplitudes, we could not have obtained this last result by superposition. It appears to be an interesting property of plane wave solutions of Maxwell's equations.

9.6 Reflection at Metallic Boundaries

This topic will be treated through the medium of Tutorial 3.

Chapter 10

RETARDED POTENTIALS

10.1 Introduction

The material in this chapter does not in a formal sense fall within the course, and will not be examined. It is included firstly to give to those students who may be interested a preview of some of the theoretical developments which are to occur in higher levels of the course, and secondly because the concepts to be presented may be appealing through their being at once simple and powerful.

10.2 Static Potentials

In previous chapters we have established that the electrostatic and magnetostatic fields at a point P_2 with position vector $\mathbf{r}_2$, arising from a distribution of charges ρ and currents $\boldsymbol{\mathcal{J}}$ at points P_1 with varying position vectors $\mathbf{r}_1$ distributed over a volume v in free space may be derived from a scalar potential ϕ and a vector potential $\boldsymbol{\mathcal{A}}$ respectively. We restate the results in equations 10.1 and 10.2 below.

$$\phi(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}_1)}{r_{12}}dv \qquad (10.1)$$

$$\boldsymbol{\mathcal{A}}(\mathbf{r}_2) = \frac{\mu_0}{4\pi}\int_v \frac{\boldsymbol{\mathcal{J}}(\mathbf{r}_1)}{r_{12}}dv \qquad (10.2)$$

In the above equations all variables are real time-varying quantities. The above integral is over the entire region occupied by the charges and currents. In terms of these potentials the electrostatic field and magnetostatic flux density are given by:

$$\boldsymbol{\mathcal{E}} = -\mathbf{grad}\,\phi \qquad (10.3)$$

$$\boldsymbol{\mathcal{B}} = \mathbf{curl}\,\boldsymbol{\mathcal{A}} \qquad (10.4)$$

As we are in free space, the magnetostatic field $\boldsymbol{\mathcal{H}}$ is obtained by dividing $\boldsymbol{\mathcal{B}}$ by μ_0.

10.3 Time Varying Fields

As indicated earlier the expressions given above for scalar and vector potentials are sufficient for the evaluation of electro*static* and magneto*static* fields. The question of what significance they may have in the calculation of time varying fields has not yet been addressed.

It is clear that these potentials will not suffice for the calculation of an electrodynamic field, as they predict a purely source-type electric field, making no allowance for the vortices of electric field which arise from the time rate of change of magnetic flux density as expressed by Faraday's law.

$$\mathbf{curl}\,\boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{10.5}$$

But there remains the hope (if you like potentials) that some other set of potentials might still suffice for the calculation of the electromagnetic field in the fully electrodynamic context.

10.4 Propagation Time Effects

There is another aspect in which we might find equations 10.1 and 10.2 unsatisfactory for dynamic fields. In the form given, they imply that the effects of charge and current density at the point P_1 are felt *immediately* at the point P_2, and make no allowance for any time delay in the transmission of those effects. Although this may be an acceptable view in the electrostatic or magnetostatic context, where nothing is changing, it is at variance with the relativity principle which states that it is impossible to transmit signals at a speed greater than the velocity of light. Some amendment to the equations should therefore be introduced to take account of this fact.

It turns out that a simple amendment which retains the basic form of the potentials but allows for their propagation at the speed of light from the source point to the field point provides the correct answer, ie an new set of potentials, called the *retarded potentials,* which allow the calculation of the electromagnetic field in the fully dynamic case via equations which are generalisations of equations 10.3 and 10.4.

10.5 The Retarded Potentials

We take the view that in the time-varying case a scalar potential ϕ and a vector potential $\boldsymbol{\mathcal{A}}$, both at a field point P_2 at time t, have broadly the from given in the electrostatic and magnetostatic equations, but because these potentials have propagated outward from the source point P_1 at the velocity of light, the values of ρ and $\boldsymbol{\mathcal{J}}$ which are to be placed under the integral signs in equations 10.1 and 10.2 are not those applying at time t but are those applying at an earlier time $t - r_{12}/c$, where r_{12} is the scalar distance between points P_1 and P_2. Thus the equations for the retarded potentials are:

$$\phi(\mathbf{r}_2, t) = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(\mathbf{r}_1, t - r_{12}/c)}{r_{12}} dv \tag{10.6}$$

$$\boldsymbol{\mathcal{A}}(\mathbf{r}_2, t) = \frac{\mu_0}{4\pi}\int_v \frac{\boldsymbol{\mathcal{J}}(\mathbf{r}_1, t - r_{12}/c)}{r_{12}} dv \tag{10.7}$$

10.6 Derivation of Fields

We must now examine the question of what new equations must be introduced to express the electric and magnetic fields from these potentials. The results to be presented will be supported in part by plausibility arguments but not by formal analytical proof. In any case it is upon their continued success in predicting empirically verifiable results that their validity ultimately depends.

In the first place, since div$\boldsymbol{\mathcal{B}} = 0$ in both the electrostatic and electrodynamic cases, we require for the calculation of the magnetic flux density only a vector potential, and there is the hope that the vector potential defined above will suffice. This is indeed confirmed by experiment, and we have

$$\boldsymbol{\mathcal{B}} = \mathbf{curl}\,\boldsymbol{\mathcal{A}} \tag{10.8}$$

where $\boldsymbol{\mathcal{A}}$ is given by equation 10.7 above. For the calculation of the electric field, as has already been remarked, some amendment of equation 10.3 is clearly necessary to allow for the newly introduced vortices of $\boldsymbol{\mathcal{E}}$. It is easy to see that a satisfactory addition is that appearing in the equation

$$\boldsymbol{\mathcal{E}} = -\mathbf{grad}\,\phi - \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t} \tag{10.9}$$

where $\boldsymbol{\mathcal{A}}$ and ϕ are the retarded scalar and vector potentials given in equations 10.6 and 10.7 above. Taking the **curl** of each side and noting that, as always with partial derivatives, the partial derivatives with respect to time and space variables commute, we obtain $\mathbf{curl}\,\boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial \mathbf{t}}$, which shows that we have indeed through the term added in equation 10.9 introduced the correct value of the vorticity to $\boldsymbol{\mathcal{E}}$.

10.7 Treasure Uncovered

We should realise that the results just produced are of quite profound significance, in that they represent a complete solution of the radiation problem when the sources are known. When an arbitrary but known charge and current distribution is present in space, the potentials can be calculated from equations 10.6 and 10.7, and then the electric field and magnetic flux density may be calculated from equations 10.8 and 10.9.

10.8 Relations Between Potentials

The electromagnetic field $\boldsymbol{\mathcal{E}}$ and flux density $\boldsymbol{\mathcal{B}}$ derivable from ϕ and $\boldsymbol{\mathcal{A}}$ are not independent. In fact among the six cartesian co-ordinate components of those vectors, equation 10.5 shows that three of these may be found from the other three, and thus only three are independent.

This fact suggests that among the three components of the vector $\boldsymbol{\mathcal{A}}$ and the single value of the scalar ϕ, there may be a relation so that only three of these four quantities are independent. This expectation is re-inforced by the realisation that ϕ and $\boldsymbol{\mathcal{A}}$ are derived respectively from ρ and $\boldsymbol{\mathcal{J}}$ and that those variables are not (in the dynamic case) independent: they are related by the conservation equation

$$\text{div}\,\boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{10.10}$$

If these matters are pursued, although we shall not present the details here, it is possible to establish that the scalar and vector potentials are related by the equation

$$\text{div}\,\boldsymbol{\mathcal{A}} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0 \tag{10.11}$$

This equation is known as the *Lorenz gauge condition.* The form quoted above is in the time domain. In the frequency domain, the result becomes

$$\text{div}\,\mathbf{A} + j\,\omega\,\mu_0\,\epsilon_0\,\phi. = 0 \tag{10.12}$$

In the form just quoted the Lorenz gauge condition provides a practical benefit. Once we have performed the integral in equation 10.7 to calculate $\boldsymbol{\mathcal{A}}$, we can avoid performing the integral in equation 10.6 to calculate the scalar potential by calculating ϕ directly from equation 10.12.

Appendix A

REFERENCES

Two text books with a view sympathetic to that presented in this course are:

1. M. N. O. Sadiku,"Elements of Electromagnetics", Saunders Publishing, (1989).

2. W. H. Hayt,"Engineering Electromagnetics", 5th edition McGraw Hill, (1989).

Material on demagnetising factors can be found in:

- J. A. Osborn, "Demagnetising Factors of the General Ellipsoid", Physical Review, vol 67, pp 351, (1945).

Appendix B

VECTOR OPERATORS IN POLAR CO-ORDINATES

In spherical polar co-ordinates at point $P(r, \theta, \phi)$ the gradient of a scalar V, the divergence of a vector $\boldsymbol{\mathcal{D}}$, and the curl of a vector $\boldsymbol{\mathcal{H}}$ are given by

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \tag{B.1}$$

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \frac{1}{r^2}\frac{\partial[r^2 D_r]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[D_\theta \sin\theta]}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \tag{B.2}$$

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{H}} = & \frac{1}{r\sin\theta}\left[\frac{\partial[H_\phi \sin\theta]}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\mathbf{a}_r \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial[rH_\phi]}{\partial r}\right]\mathbf{a}_\theta \\ & + \frac{1}{r}\left[\frac{\partial[rH_\theta]}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\mathbf{a}_\phi \end{aligned} \tag{B.3}$$

Appendix C

SUMMARY OF BOUNDARY CONDITIONS

SUMMARY OF BOUNDARY CONDITIONS				
Time variation	General case	Intermediate conductivity	Perfect insulator	Very good conductor
dc	$\boldsymbol{\mathcal{E}}_t$ is cts			$\boldsymbol{\mathcal{E}}_t = 0$
ac	$\boldsymbol{\mathcal{E}}_t$ is cts			$\boldsymbol{\mathcal{E}}_t = 0$
dc	$\hat{\mathbf{n}} \times (\boldsymbol{\mathcal{H}}_{t2} - \boldsymbol{\mathcal{H}}_{t1}) = \boldsymbol{\mathcal{J}}_s$	$\boldsymbol{\mathcal{J}}_s = 0$	$\boldsymbol{\mathcal{J}}_s = 0$	
ac	$\hat{\mathbf{n}} \times (\boldsymbol{\mathcal{H}}_{t2} - \boldsymbol{\mathcal{H}}_{t1}) = \boldsymbol{\mathcal{J}}_s$	$\boldsymbol{\mathcal{J}}_s = 0$	$\boldsymbol{\mathcal{J}}_s = 0$	$\hat{\mathbf{n}} \times \boldsymbol{\mathcal{H}}_t = \boldsymbol{\mathcal{J}}_s$
dc	$\boldsymbol{\mathcal{D}}_{n2} - \boldsymbol{\mathcal{D}}_{n1} = \rho_s$			$\boldsymbol{\mathcal{D}}_n = \rho_s$
ac	$\boldsymbol{\mathcal{D}}_{n2} - \boldsymbol{\mathcal{D}}_{n1} = \rho_s$		$\rho_s = 0$	$\boldsymbol{\mathcal{D}}_n = \rho_s$
dc	$\boldsymbol{\mathcal{B}}_n$ is cts			
ac	$\boldsymbol{\mathcal{B}}_n$ is cts			$\boldsymbol{\mathcal{B}}_n = 0$

PAGE

Appendix D

EXERCISES

D.1 Exercise 1

1. Given two static vector fields (i) $\boldsymbol{\mathcal{E}} = \mathbf{u}_x x$, and (ii) $\boldsymbol{\mathcal{H}} = \mathbf{u}_x y$:-

   (a) provide a simple description in words of each field;

   (b) classify the fields by taking the divergence and curl of both vectors; and

   (c) assuming there is neither polarisation nor magnetisation present, determine the distribution of charges and currents which must accompany each of the fields.

2. For the potential field given in free space in spherical polar co-ordinates as V = 100r Volts, find:

   (a) the electric field;

   (b) the electric flux density; and

   (c) the amount of charge lying within the sphere r = 0.5m.

D.2 Exercise 2

1. An inner sphere of diameter 10 mm constructed from material of dielectric permittivity 20pF/m has an electric conduction charge of 1nC uniformly distributed within its volume.

   That sphere is surrounded by an uncharged concentric thick spherical shell of inner diameter 20mm and outer diameter 30mm made of the same material but carrying no conduction charge.

   All other regions contain empty space.

   Determine at all points of space expressions for:

   (a) the electric field $\boldsymbol{\mathcal{E}}$;

   (b) the electric flux density $\boldsymbol{\mathcal{D}}$;

   (c) the polarisation $\boldsymbol{\mathcal{P}}$;

(d) the induced volume charge density ρ^i;

and determine at each of the three material boundaries expressions for:

(e) the induced surface charge density σ^i.

2. Two perfectly conducting surfaces are located at $r = 2$ cm and $r = 10$ cm. The total current passing radially outward through the medium between the spheres is 2.5A.

   (a) Find the voltage and resistance between the spheres, and the electric field between the spheres, if a conducting material of conductivity 20 mS/m is placed between them.

   (b) Find the voltage between the spheres and the electric field between the spheres if the material between them has a conductivity of 1/r mS/m, where r is in metres.

   (c) What can be said in each case about the charge densities ρ^c, ρ^i, and ρ^t between the spheres? In each case a uniform dielectric constant of 2.5 may be assumed.

D.3 Exercise 3

The electric field intensity associated with a plane electromagnetic wave approaching in vacuum the boundary z = 0 of a perfect conductor is given in SI units by the expression:

$$\boldsymbol{\mathcal{E}}(x, y, z, t) = 10 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos[2\pi(ft - z)]\text{Vm}^{-1} \tag{D.1}$$

Determine:

1. an expression as a function of position and time for the associated magnetic field;
2. the frequency f of the wave;
3. the power carried by the wave; and
4. an expression as a function of position and time for the real surface current density set up by the wave at its reflection in the plane $z = 0$.

D.4 Exercise 4

The parallel plate capacitor shown in Figure D.1 has top and bottom plates 120 mm square and is partly filled with a dielectric of dielectric constant 2.25 and area equal to the plates, the remaining space being occupied by air. A voltage of 10 volts is established and maintained between the plates by means of the battery shown.

1. Calculate in an appropriate order

   - the electric field in the regions A, B and C;

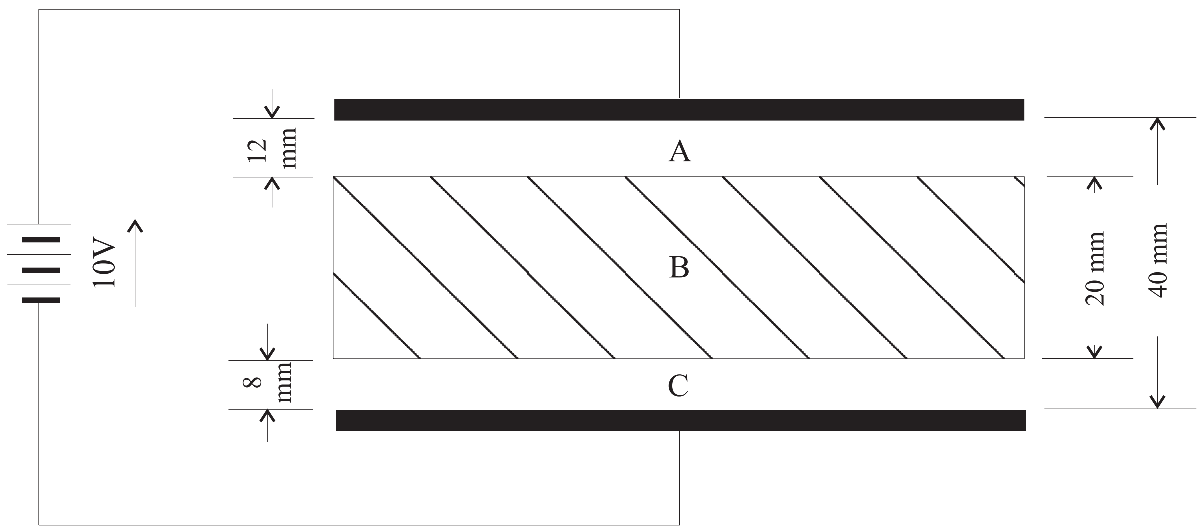


Figure D.1: Layered-dielectric parallel-plate capacitor.

- the electric flux density in each of those regions;
- the surface charge density on each of the plates; and
- the induced surface charge density on each of the dielectric surfaces.

2. The dielectric slab is withdrawn from between the plates by pulling it horizontally. Calculate the change in stored energy in the capacitor, the energy supplied by the battery, and the mechanical work done in the withdrawal.

D.5 Exercise 5

D.5.1 Objectives

The objectives pursued in this problem are

- To give practice in calculating a magnetic field from a simple current distribution.
- To deepen understanding of the relationships between internal and external fields in a magnetic material.
- To re-inforce the concept of *depolarising and demagnetising factors* by means of which, for ellipsoidal shapes, the relation between the depolarising or demagnetising fields which arise from induced surface electric charge or magnetic pole densities, the material shape, and the internal polarisation and magetisation, may be expressed.

In order to simplify the calculations required in these problems, the magnetic material *yttrium iron garnet* which has some unusual properties which allow such simplification, is introduced.

D.5.2 Preamble

Depolarising and demagnetising factors are dimensionless numbers which enable the calculation, for ellipsoidal shapes of uniformly polarised or magnetised dielectric or magnetic material, the contributions to the internal electric or magnetic fields arising from the surface densities of electric or magnetic charges produced at the surfaces of such ellipsoids by the discontinuity of polarisation or magnetisation occurring there.

In an xyz coordinate system directed along the principal axes of the ellipsoid, the components of the depolarising field $\mathbf{E}_d$ or demagnetising field $\mathbf{H}_d$ are given in terms of the components of the polarisation and magnetisation by the matrix relations

$$\begin{bmatrix} E_{dx} \\ E_{dy} \\ E_{dz} \end{bmatrix} = -\frac{1}{\epsilon_0} \begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & Z_z \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \tag{D.2}$$

and

$$\begin{bmatrix} H_{dx} \\ H_{dy} \\ H_{dz} \end{bmatrix} = - \begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & Z_z \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \tag{D.3}$$

where N_z, N_y and N_z are the *depolarising or demagnetising factors* along the principal axes of the ellipsoid.

Note in the above equations the minus signs, which derive from the fact that the depolarising or demagnetising fields *oppose* the polarisation or magnetisation in the material and from the convention that the depolarising or demagnetising factors are quoted as positive numbers.

The depolarising or demagnetising factors are very complicated functions of the shape factors of the ellipsoid, but it can be shown that they always satisfy the relation

$$N_x + N_y + N_z = 1 \tag{D.4}$$

This relation, together with the application of basic principles enables the determination of depolarising or demagnetising factors for ellipsoids of particular symmetry or extreme shape.

D.5.3 Question on Depolarising or Demagnetising Factors

Insert into Figure D.2 the values of depolarising or demagnetising factors along the principal axes for the shapes shown.

D.5.4 Field at Centre of Single Turn Coil

Calculate the magnetic field H at the centre of a *large circular loop* of radius R of one turn or wire carrying a steady current I. What is the direction of this field?

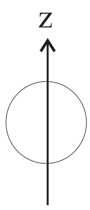
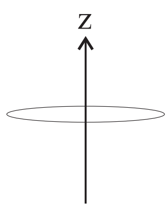
SHAPE	AXES	N_x	N_y	N_z
Sphere				
Long rod				
Thin disc				

Figure D.2: Demagnetising factors for various shapes.

D.5.5 Minimum Current for Saturation of a YIG Sphere

The compound *ytrrium iron garnet*, commonly known as YIG, has in its single crystal form a *saturation magnetisation* M of 138,000 $\mathrm{Am^{-1}}$ and an extremely low *coercive force* of 30 $\mathrm{Am^{-1}}$. Thus it is possible to magnetise it to *complete saturation* by the application of a negligibly small internal magnetic field.

The material is of importance in electrical engineering applications because it can be made to exhibit *gyromagnetic resonance* with extremely low loss. Small spheres of the material immersed in a combination of a steady magnetic field from a solenoid, and a high frequency magnetic field from a transmission line or waveguide, are used to make narrow-band low-loss electronically-tunable high-frequency filters.

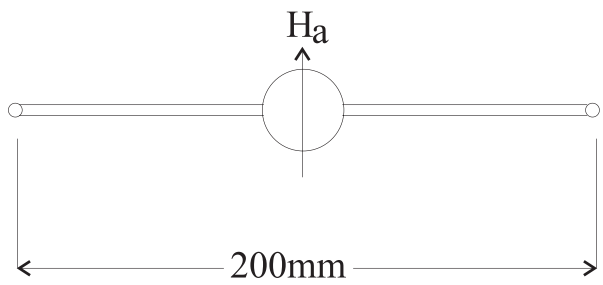


Figure D.3: Sphere in single turn coil.

A small sphere if YIG of diameter 1 mm is placed at the centre of a single-turn circular coil of diameter 200 mm as shown in Figure D.3. Calculate the current in the wire required just to produce magnetic saturation of the sphere.

D.5.6 Field Distributions

At that current sketch in the spaces provided in Figure D.4 the distribution inside and outside the sphere of

- the applied field $\mathbf{H}_a$;
- the magnetisation $\mathbf{M}$;
- the demagnetising field $\mathbf{H}_d$;
- the total magnetic field $\mathbf{H}$;
- the magnetic flux density $\mathbf{B}$;

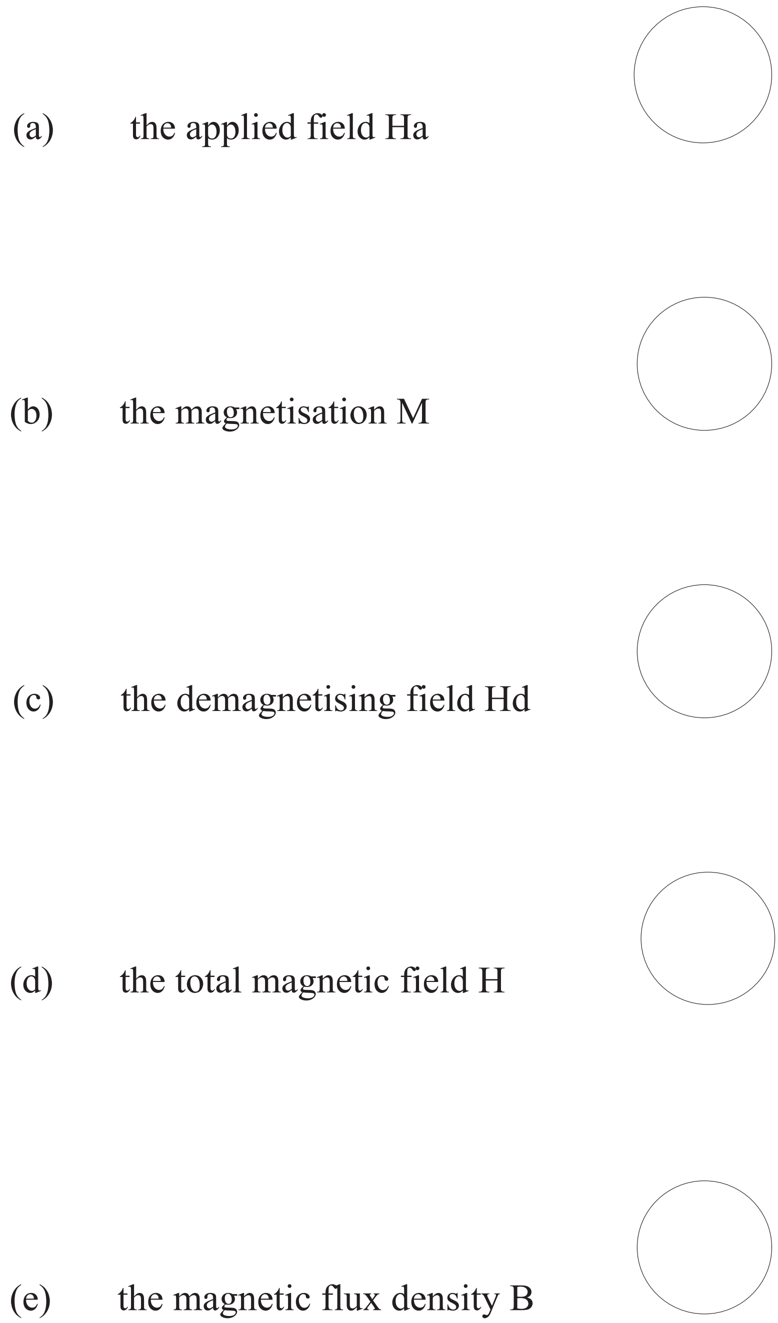


Figure D.4: Field distributions at onset of saturation.

Appendix E

ANSWERS TO EXERCISES

E.1 Exercise 1

1. **Exercises on Vector Fields**

   (a) (i) This is a field which is directed always along the x axis, is independent of the y and z co-ordinates, is zero in the plane $x = 0$, and increases linearly in magnitude with x.

   (ii) This is a field which is directed always along the x axis, is independent of the x and z co-ordinates, is zero in the plane $y = 0$, and increases linearly in magnitude with y.

   (b) (i) As $\nabla\cdot\boldsymbol{\mathcal{E}} = 1Vm^{-2}$ and $\nabla\times\boldsymbol{\mathcal{E}} = 0$, this field is purely *source type* (irrotational).

   (ii) As $\nabla\cdot\boldsymbol{\mathcal{H}} = 0$ and $\nabla\times\boldsymbol{\mathcal{H}} = -\hat{\mathbf{u}}_z Am^{-2}$, this field is purely *vortex type* (solenoidal).

   (c) (i) This is an electrostatic field with $\nabla \cdot \boldsymbol{\mathcal{D}} = \epsilon \times 1Vm^{-2}$. Assuming free space, $\epsilon = \epsilon_0 = 8.854\,pFm^{-1}$. Thus a volume charge density of $8.854\,pCm^{-3}$ would be required to accompany such a field. Note that this charge density does not accompany this field to the exclusion of others; we could, with the aid of sources outside the region for which the sources and vortices were specified, produce other fields which have the same divergence and curl within the specified region.

   (ii) We might assume for convenience this is a magnetostatic field, ie it is produced by currents rather than changing electric flux density. Since it has zero divergence, there appears to be no magnetic material involved; at least there is no divergence of $\boldsymbol{\mathcal{M}}$. To support the **curl** of $\boldsymbol{\mathcal{H}}$ we require a volume current density of $-1\,Am^{-2}$ directed along the z axis.

2. **Exercises on Potential Fields**

   The potential is given as $V = 100r\,V$. In spherical polar co-ordinates **grad**V has the components

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{a}}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\mathbf{a}}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\mathbf{a}}_\phi \tag{E.1}$$

   Since $\boldsymbol{\mathcal{E}} = -\nabla V$ and there is no θ or ϕ dependence we have

$$\boldsymbol{\mathcal{E}} = -100\,\hat{\mathbf{a}}_r\,Vm^{-1} \tag{E.2}$$

This is a field which points normally inwards towards the origin, and of constant magnitude. Note however that it is not a uniform field. The flux density $\boldsymbol{\mathcal{D}}$ is just ϵ_0 times the above $\boldsymbol{\mathcal{E}}$.

To find the charge density ρ we must evaluate $\nabla \cdot \boldsymbol{\mathcal{D}}$. Again in spherical polar co-ordinates this is

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(D_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \tag{E.3}$$

Again as there is no θ or ϕ dependence, only the first term counts and we have

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \frac{-100\,\epsilon_0}{r^2}\frac{\partial(r^2)}{\partial r} \tag{E.4}$$

ie

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \frac{-200\,\epsilon_0}{r}\,Cm^{-3} \tag{E.5}$$

This is the charge density ρ in Cm^{-3}. To obtain the total charge within a sphere of diameter r we integrate from 0 to r, taking note that a volume element $dv = 4\pi r^2 dr$. Thus

$$Q(r) = \int_0^r -4\pi r^2 200\epsilon_0 r^{-1} dr \tag{E.6}$$

ie

$$Q(r) = -400\pi\epsilon_0 r^2 \tag{E.7}$$

If we substitute the value $r = 0.5$ we obtain

$$Q(0.5) = -2.78\,\mathrm{nC} \tag{E.8}$$

E.2 Exercise 2

1. **Dielectric Sphere Problem**

   We can most simply solve first for the $\boldsymbol{\mathcal{D}}$ vector as this depends only on the conduction charges, which are located within the centre sphere, and does not depend upon any of the induced charges which are located either within or on the surface of the dielectric media. For a spherical volume of radius r with $0 < r < 5mm$, the enclosed conduction charge is

$$Q(r) = 10^{-9}\left[\frac{r}{5mm}\right]^3 \tag{E.9}$$

ie $8 \times 10^{-3} r^3\, C$. This charge must be equal to the flux of $\boldsymbol{\mathcal{D}}$ outward from the surface, ie

$$4\pi r^2 D_r = 8 \times 10^{-3} r^3 \tag{E.10}$$

Thus for $0 < r < 5 \times 10^{-3}\, m$

$$D_r = \frac{2}{\pi} \times 10^{-3} r \tag{E.11}$$

For any radius r greater than $5mm$ the enclosed conduction charge remains constant at $1nC$. Thus

$$D_r = \frac{10^{-9}}{4\pi r^2} \tag{E.12}$$

In both the above expressions, the units of D_r are Cm^{-2}.

For any one of the four regions shown in the figure, the radial component of the electric field is related to D_r by

$$E_r = D_r / \epsilon \tag{E.13}$$

where ϵ is either $\epsilon_1 = 20\, pFm^{-1}$ or is $\epsilon_0 = 8.854\, pFm^{-1}$.

For the two dielectric regions the radial component of polarisation is given by

$$P_r = (\epsilon_1 - \epsilon_0) E_r \tag{E.14}$$

ie

$$P_r = (\frac{\epsilon_1 - \epsilon_0}{\epsilon_1}) D_r \tag{E.15}$$

Inside the volume of the dielectric the induced volume charge density ρ_v^i is given by

$$\rho_v^i = -\text{div}\, \boldsymbol{\mathcal{P}} \tag{E.16}$$

In a purely radially directed field the divergence of $\boldsymbol{\mathcal{P}}$ in spherical polar co-ordinates becomes simply

$$\text{div}\, \boldsymbol{\mathcal{P}} = \frac{1}{r^2} \frac{\partial (r^2 P_r)}{\partial r} \tag{E.17}$$

Finally the induced surface charge density at an interface between polarisation and free space is given by

$$\rho_s^i = \boldsymbol{\mathcal{P}} \cdot \mathbf{n} \tag{E.18}$$

where $\mathbf{n}$ is an outward unit normal vector. If we now apply these principles in turn to each of the four regions shown in the figure we obtain

(a) For $0 < r < 5 \times 10^{-3}\, m$

$$
\begin{aligned}
E_r &= 3.183 \times 10^7 r V m^{-1} && \text{(E.19)}\\
D_r &= 6.366 \times 10^{-4} r C m^{-2} && \text{(E.20)}\\
P_r &= 3.548 \times 10^{-4} r C m^{-2} && \text{(E.21)}\\
\rho_v^i &= -1.064 \times 10^{-3} C m^{-3} && \text{(E.22)}\\
\rho_s^i &= 1.774 \times 10^{-6} C m^{-2} \text{on surface} && \text{(E.23)}
\end{aligned}
$$

(b) For $5 \times 10^{-3} < r < 10 \times 10^{-3}\, m$

$$
\begin{aligned}
E_r &= 8.988 r^{-2} V m^{-1} && \text{(E.24)}\\
d_r &= 7.958 \times 10^{-11} r^{-2} C m^{-2} && \text{(E.25)}\\
P_r &= 0 && \text{(E.26)}\\
\rho_v^i &= 0 && \text{(E.27)}\\
\rho_s^i &= \text{see part (c) below} && \text{(E.28)}
\end{aligned}
$$

The same results apply to the region $r > 15\, mm$.

(c) For $10 \times 10^{-3} < r < 15 \times 10^{-3}\, m$

$$
\begin{aligned}
E_r &= 3.979 r^{-2} V m^{-1} && \text{(E.29)}\\
D_r &= \text{as for part (b) above} && \text{(E.30)}\\
P_r &= 44.35 \times 10^{-12} r^{-2} C m^{-2} && \text{(E.31)}\\
\rho_v^i &= 0 && \text{(E.32)}\\
& && \text{(E.33)}
\end{aligned}
$$

On the inner surface where $P_r = 4.435 \times 10^{-7}\, Cm^{-2}$, the induced surface charge density is this value with the sign changed. On the outer surface where $P_r = 1.971 \times 10^{-7}\, Cm^{-2}$, the induced surface charge density is this value with the sign preserved.

2. **Conducting Shell Problem**

In this problem the $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{P}}$ fields and the current density $\boldsymbol{\mathcal{J}}$ will be all radial. The radial component $\boldsymbol{\mathcal{J}}_r$ of the volume current density will adjust itself so that the total current crossing and spherical surface of radius r between the conducting boundaries will be constant. Thus

$$4\pi r^2 J_r = 2.5\, A \qquad \text{(E.34)}$$

ie

$$J_r = \frac{2.5}{4\pi} r^{-2}\, Am^{-2} \qquad \text{(E.35)}$$

The above part of the answer will be common to both cases discussed below.

(a) **Case of Uniform Conductivity** $\sigma = 20\, mSm^{-1}$
Since $\boldsymbol{\mathcal{J}} = \sigma\, \boldsymbol{\mathcal{E}}$ we have

$$E_r = \frac{2.5 \times 10^3}{4\pi \times 20} r^{-2} \tag{E.36}$$

ie

$$E_r = 9.947\, r^{-2}\, Vm^{-1} \tag{E.37}$$

The potential of the outer sphere of radius $b = 100\, mm$ relative to the inner sphere of radius $a = 20mm$ is given by

$$V_{ab} = -\int_a^b E_r(r)dr \tag{E.38}$$

ie $V_{ab} = -397.8V$. The potential of the inner sphere relative to the outer sphere is $397.8V$. The resistance R between the spheres is $R = 3.798V/2.5A$, ie 159.2Ω

(b) **Case of Non-uniform Conductivity**
When $\sigma = 10^{-3}/rSm^{-1}$ we have $E_r = J_r/\sigma = 189.9r^{-1}Vm^{-1}$. Then

$$V_{ab} = -198.9 \int_a^b \frac{dr}{r} \tag{E.39}$$

which evaluates to $-320.2V$. The potential of the inner sphere relative to the outer sphere is thus $320.2V$. The resistance between the spheres is $R = 320.2V/2.5A$ which comes to $128.1\,\Omega$.

(c) **Discussion of Charge Density**
As indicated above the current density $\boldsymbol{\mathcal{J}}$, which is driven by the electric field $\boldsymbol{\mathcal{E}}$, will adjust itself to give a total current I which is equal to $2.5A$ at all radii. Such an obliging electric field may or may not be provided by charges only on the inner and outer conducting surfaces. There may be a need for an additional volume charge density in the space between these surfaces. Whatever the case, we will still have everywhere in that space the relations $\boldsymbol{\mathcal{D}} = \epsilon_r\epsilon_0\, \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{P}} = (\epsilon_r - 1)\epsilon_0\, \boldsymbol{\mathcal{E}}$. To support these fields we may need volume charge densities ρ_v^c, ρ_v^i, and ρ_v^t, respectively given by

$$\rho_v^c = \nabla \cdot \boldsymbol{\mathcal{D}} \tag{E.40}$$
$$\rho_v^i = -\nabla \cdot \boldsymbol{\mathcal{P}} \tag{E.41}$$
$$\rho_v^t = \epsilon_0 \nabla \cdot \boldsymbol{\mathcal{E}} \tag{E.42}$$

Now applying these principles to case (a) above, we note that since E_r has an r^{-2} variation it has no divergence, so all of the above charge densities are zero. In case (b), however, we have $E_r = 198.9\, r^{-1}$, ie $\epsilon_0\, E_r = 1.7611 \times 10^{-9} r^{-1} Cm^{-2}$. P_r will be 1.5 times this value and D_r will be 2.5 times this value. Performing the divergence operation in spherical polar co-ordinates gives

$$\rho_v^c = 4.403 \times 10^{-9} r^{-2}\, Cm^{-3} \tag{E.43}$$
$$\rho_v^i = -2.642 \times 10^{-9} r^{-2}\, Cm^{-3} \tag{E.44}$$
$$\rho_v^t = 1.761 \times 10^{-9} r^{-2}\, Cm^{-3} \tag{E.45}$$

E.3 Exercise 3

1. In free space for a propagating wave the electric and magnetic fields are in time phase, are mutually orthogonal and are orthogonal to the propagation direction, with $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{H}}$ and $\vec{\beta}$ (in that order) forming a right hand system.

   Since in this case the wave is propagating in the $+z$ direction and the electric field is in the $+y$ direction we expect the magnetic field to be in the $-x$ direction to make a right hand system. The peak value of the magnetic field will be $(10/\eta)\, Am^{-1}$ where $\eta = 377\,\Omega$. Thus

$$\boldsymbol{\mathcal{H}}(x, y, z, t) = \frac{10}{377} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos[2\pi(ft - z)] Am^{-1} \tag{E.46}$$

   We will obtain a particular value for f in the next section.

2. The general form of the position and time dependence for a sinusoidal travelling wave with propagation constant β in the z direction is $\cos(\omega t - \beta z)$. Comparing this form with that given, we see that $\beta = 2\pi\, m^{-1}$. The wave length λ is thus one metre.

   The relation between frequency, wave length and velocity being $v = f\lambda$, and the velocity of electromagnetic waves in free space being $3 \times 10^8\, ms^{-1}$, we conclude the frequency f is 300 MHz.

3. The electric and magnetic fields of the wave can be represented by complex phasors (which represent peak not rms values) $E_y = 10e^{-j\beta z} Vm^{-1}$ and $H_x = (-10/377)e^{-j\beta z}\, Am^{-1}$. From these phasors we see the complex Poynting vector has only a z component

$$S_z = -\frac{1}{2} E_y H_x^* \tag{E.47}$$

   ie

$$S_z = \frac{50}{377}\, Wm^{-2} \tag{E.48}$$

   This number is purely real and gives the power per unit area carried by the wave in the $+z$ direction.

4. When such a wave encounters at right angles a perfectly conducting plane, an equal-amplitude oppositely-directed reflected wave is set up. The phase relations between

the fields of the incident wave and those of the reflected wave are such that the electric fields exactly *cancel* and the magnetic fields *double*.

Thus in the plane $z = 0$ we expect to have a magnetic field

$$\boldsymbol{\mathcal{H}}(x, y, 0, t) = \frac{20}{377}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\cos(2\pi f t)\ Am^{-1} \tag{E.49}$$

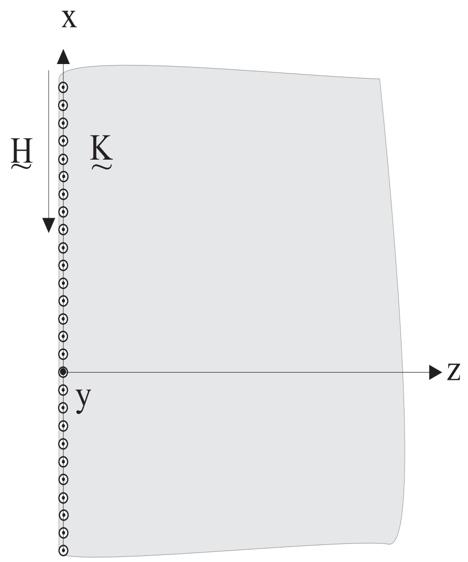


Figure E.1: Surface current at a metal boundary

When such a magnetic field exists adjacent to the boundary of a perfect conductor there is required a surface current density equal in magnitude and in time phase with the field, but in the orthogonal direction, with the sense determined by the right hand rule, as shown in Figure E.1.

Thus a surface current density in the $+y$ direction will produce on the $z < 0$ side of the xy plane a magnetic field in the $-x$ direction. Thus the expression for the surface current density is

$$\boldsymbol{\mathcal{K}}(x, y, 0, t) = \frac{20}{377}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\cos(2\pi f t)\ Am^{-1} \tag{E.50}$$

E.4 Exercise 4 and 5

Still to come, except for the fragment below.

E.4.1 Question on Depolarising or Demagnetising Factors

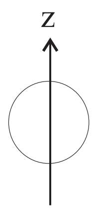
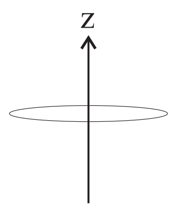
SHAPE	AXES	N_x	N_y	N_z
Sphere		1/3	1/3	1/3
Long rod		1/2	1/2	0
Thin disc		0	0	1

Figure E.2: Demagnetising factors for various shapes.

Insert into Figure E.2 the values of depolarising or demagnetising factors along the principal axes for the shapes shown.