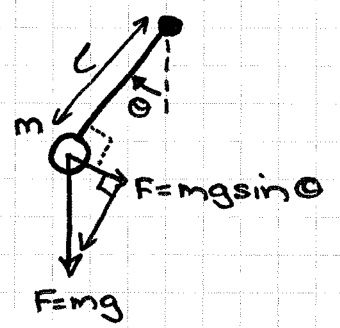
5mm Squares

# 12R012 - PENDULUM MODELLING AND SIMULATION 25-FEB-12

**Aim** - describe the modelling and simulation of a pendulum

## 1. VERTICAL PENDULUM MODEL

model a simple pendulum as a mass m, at the end of a massless rod of length $\ell$




restoring torque T on pendulum is

$$T = F \times \ell = -mg\ell\sin\theta$$

↖ restoring torque is opposite to displacement

moment of inertia $J = m\ell^2$

angular acceleration $\alpha$

$$\alpha = \frac{T}{J} = \frac{-mg\ell\sin\theta}{m\ell^2}$$

$$= -\frac{g}{\ell}\sin\theta$$

$$\text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

is the differential equation which describes its motion

## 2. SIMULATION

### 2.1. Using Excel

the pendulum has two state variables: position $\theta$, and angular velocity $\omega$

we can write equations for the rate of change of these state variables:

$$\frac{d\omega}{dt} = \alpha = -\frac{g}{\ell}\sin\theta$$

$$\frac{d\theta}{dt} = \omega$$

in discrete time, these can be written as the values at time step k as a function of the values at time step k-1

$$\alpha(k) = -g/\ell \times \sin[\theta(k-1)]$$

$$\omega(k) = \omega(k-1) + \alpha(k) \times T$$

$$\theta(k) = \theta(k-1) + \omega(k) \times T$$

where T is the time-step period

this is basically Euler integration where:

$$\frac{d\omega}{dt} \approx \frac{\omega(k) - \omega(k-1)}{T} = \alpha(t)$$

the above equations can be solved in Excel assuming some initial values for $\theta$ and $\omega$:

| | $\alpha(k)$ | $\omega(k)$ | $\theta(k)$ |
|---|---|---|---|
| k=0 | | 0 | 90° |
| k=1 | $\alpha(1)$ | $\omega(1)$ | $\theta(1)$ |
| k=2 | $\alpha(2)$ | $\omega(2)$ | $\theta(2)$ |
| k=3 | $\alpha(3)$ | $\omega(3)$ | $\theta(3)$ |
| ⋮ | ⋮ | ⋮ | ⋮ |

### 2.2. Using Simulink

Simulink uses a simple control system block diagram interface which readily allows simulation of the above differential equations

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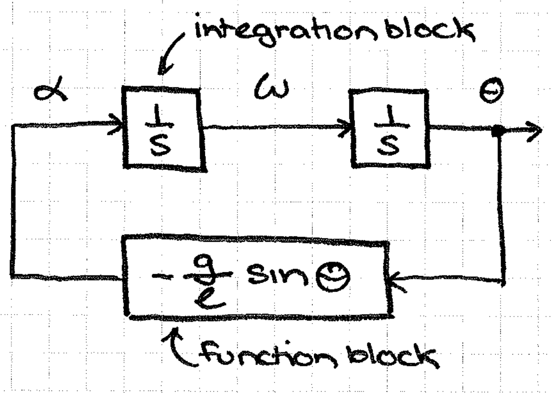


for each integration block, need to set initial conditions (values)

## 3. MODEL RESULTS

### 3.1. Analytical

now $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$

for small $\theta$, $\sin\theta \approx \theta$

thus $\frac{d^2\theta}{dt^2} = -g/\ell$

from this can show that period:

$T \doteq 2\pi\sqrt{\frac{\ell}{g}}$ (for small displacements)