Chapter 4: Equivalent Circuits

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Abstract – this section describes the theory and application of the electrical equivalent circuit for DC and surface and interior PM machines. It also covers the concepts of constant torque versus field-weakening operation and saturation.

I. Electrical Equivalent Circuits

The concept of the electrical equivalent circuit is a powerful method for analysing and obtaining intuitive insight into the performance of electrical machines. It is an essential tool which electric machine designers need to be familiar with.

The equivalent circuit seeks to provide a simplified model to represent the electrical and mechanical behaviour of the machine. The circuit contains a number of equivalent circuit parameters. These can be determined using analytical, finite-element or experimental approaches. Once these parameters have been found, the equivalent circuit can be used to predict the performance of the machine over a wide range of operation.

Equivalent circuits are widely used to model transformers, induction machines and synchronous machines such as surface permanent magnet (PM), synchronous reluctance and interior PM machines (see Fig. 1).

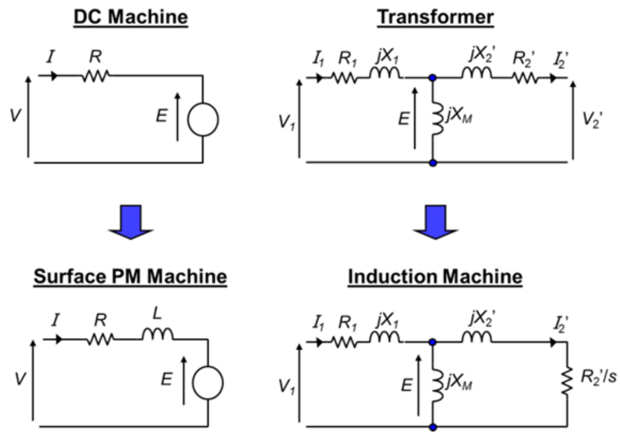

Fig. 1. Equivalent circuits for transformers and example machines.

The voltages and currents in AC equivalent circuits can also be represented as a phasor diagram. This phasor diagram is often helpful to understand the operation of the machine.

Certain machines such as interior PM and synchronous reluctance machines use two equivalent circuits to represent the fact that reactance in the *d*- and *q*-axis of the machine are different.

As equivalent circuits are electrical, the electrical to mechanical energy conversion in the machine is modelled using a circuit element. For PM machines this is a voltage source which either absorbs power in motoring operation or produces power in generating operation. In induction machines, this is a resistor which can either have positive or negative resistance value.

In general, electrical equivalent circuits contain the following components.

Resistors are used to model power being absorbed. Resistors which are in series with respect to the major current path in the equivalent circuit are used to model copper loss due to the resistance of windings. Shunt resistors are used to model iron loss (particularly eddy-current loss).

Inductors model the energy storage in magnetic fields. Series inductors are used for modelling leakage reactance and shunt inductors are used for modelling the magnetising reactance. This is shown in the induction equivalent circuit in Fig. 1.

Voltage sources appear in synchronous machine equivalent circuits and model the induced voltage in the stator winding due to the rotor magnetic field. In the induction machine the induced voltage is produced across the magnetising reactance.

II. Permanent Magnet DC Machine

The equivalent circuit for a permanent magnet (PM) DC machine (see Fig. 2) consists of a series combination of an induced (back-EMF) voltage source E and an armature resistance R which models the armature winding and brush resistances. The iron and mechanical losses of the machine are modelled as a no-load power loss $P_{nl}(\omega_m)$ which is assumed only a function of mechanical speed ω_m. Fig. 3 shows an example of a measured no-load loss versus speed curve for a machine.

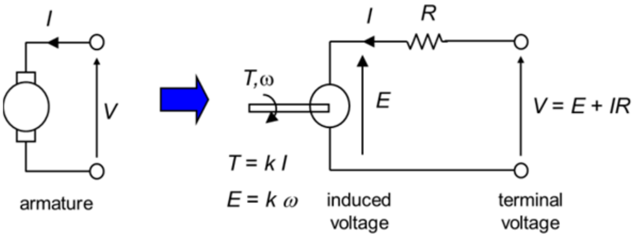

Fig. 2. Equivalent circuit for a PM DC motor.

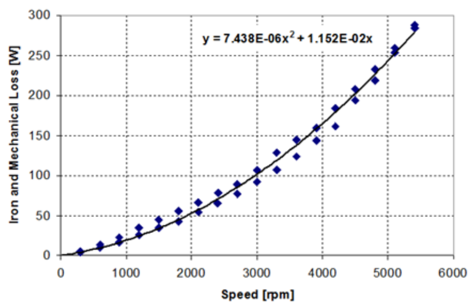

Fig. 3. Example of measured no-load loss versus speed.

It is assumed in the following that the (dc) armature current I is defined as entering the equivalent circuit (see Fig. 2) and so $I > 0$ for motors and $I < 0$ for generators.

The (dc) terminal voltage V, is given in terms of the armature current I as,

$$V = E + IR \quad [\text{V}] \tag{1}$$

The induced voltage E models the conversion of electrical power to and from mechanical power. It is equal to the product of the back-emf constant k in V/(rad/s) and the mechanical angular speed ω_m in rad/s.

$$E = k\omega_m \quad [\text{V}] \tag{2}$$

The electromagnetic torque T_{em} is equal to the product of the back-emf constant k and the armature current I,

$$T_{em} = kI \quad [\text{Nm}] \tag{3}$$

From this equation, it can be seen why the back-emf constant k is also known as the torque constant with units of Nm/A.

The electromagnetic power P_{em} is given by,

$$P_{em} = T_{em}\omega_m = EI \quad [\text{W}] \tag{4}$$

The no-load power loss, P_{nl}, can also be expressed as a no-load torque loss, T_{nl},

$$P_{nl} = T_{nl}\omega_m \quad [\text{W}] \tag{5}$$

For a motor, the no-load power (and torque) losses are subtracted from the electromagnetic power (and torque) to form the output mechanical power (and torque). The opposite occurs for generators where these losses are added to the corresponding electromagnetic quantities to give the input mechanical power (or torque),

$$P_m = \begin{cases} P_{em} - P_{nl} \quad [\text{W}] & \text{for motors} \\ P_{em} + P_{nl} \quad [\text{W}] & \text{for generators} \end{cases} \tag{6}$$

which corresponds to,

$$T_m = \begin{cases} T_{em} - T_{nl} \quad [\text{Nm}] & \text{for motors} \\ T_{em} + T_{nl} \quad [\text{Nm}] & \text{for generators} \end{cases} \tag{7}$$

The electrical power P_e is,

$$P_e = VI \quad [\text{W}] \tag{8}$$

The efficiency is given by,

$$\eta = \begin{cases} \dfrac{P_m}{P_e} & \text{for motors} \\ \dfrac{P_e}{P_m} & \text{for generators} \end{cases} \tag{9}$$

A. DC Motor Example

Consider a DC machine with a back-emf constant k of 0.1V/(rad/s) and an armature resistance R of 0.15Ω. It is desired to produce an output torque of 3 Nm at 2,000rpm. The no-load iron and mechanical loss at this speed is known to be 30W. Find the required input voltage and current, input power, output torque and power, and efficiency.

The mechanical angular speed ω_m is,

$$\omega_m = \frac{2000\text{rpm}\bullet 2\pi}{60} = 209.4\text{rad/s} \tag{10}$$

The equivalent no-load (loss) torque is given by,

$$T_{nl} = \frac{30\text{W}}{209.4\text{rad/s}} = 0.1433\text{Nm} \tag{11}$$

The required electromagnetic torque T_{em} is,

$$T_{em} = T_m + T_{nl} = 3\text{Nm} + 0.143\text{Nm} = 3.143\text{Nm} \tag{12}$$

The required armature current I is,

$$I = \frac{T_{em}}{k} = \frac{3.143\text{Nm}}{0.1\text{Nm/A}} = 31.43\text{A} \tag{13}$$

The back-emf E is,

$$E = k\omega = 0.1\text{V/(rad/s)}\bullet 209.4\text{rad/s} = 20.94\text{V} \tag{14}$$

The terminal voltage V is,

$$V = E + IR = 20.94\text{V} + 31.43\text{A}\bullet 0.15\Omega = 25.66\text{V} \tag{15}$$

The (input) electrical power P_e is,

$$P_e = VI = 25.66\text{V}\bullet 31.43\text{A} = 806.5\text{W} \tag{16}$$

The (output) mechanical power P_m is,

$$P_m = T_m\omega_m = 3\text{Nm}\bullet 209.4\text{rad/s} = 628.2\text{W} \tag{17}$$

The efficiency η is given by the output power divided by the input power,

$$\eta = \frac{P_m}{P_e} = \frac{628.2\text{W}}{806.5\text{W}} = 77.9\% \tag{18}$$

B. DC Generator Example

In this second example another DC machine is used as a generator. This DC machine is tested and under open-circuit conditions at 500rpm, the induced voltage is 15V. At the same speed, the short-circuit current is 24A. Now if a 5Ω resistor is connected to the output and the machine is spun at 800 rpm, find the output power and efficiency. The no-load iron and mechanical loss at 800 rpm is known to be 10W.

Firstly, find the equivalent circuit parameters. The back-emf constant k can be found from the induced voltage and the speed at which the open-circuit test is performed,

$$k = \frac{E}{\omega_m} = \frac{15\text{V}}{\left(\dfrac{500\text{rpm}\bullet 2\pi}{60}\right)} = 0.2865\text{V/(rad/s)} \tag{19}$$

From the equivalent circuit, the armature resistance R can be found from the induced (open-circuit) voltage and the short-circuit current I_{SC} at the same speed,

$$R = \frac{E}{I_{SC}} = \frac{15\text{V}}{24\text{A}} = 0.625\Omega \tag{20}$$

At 800rpm, the induced voltage E can be found from either the back-emf constant k or by knowing that the induced voltage is proportional to speed and scaling the value at 500 rpm,

$$\begin{aligned} E = k\omega_m &= 0.2865\text{V/(rad/s)}\bullet\frac{800\text{rpm}\bullet 2\pi}{60} \\ &= 15\text{V}\,\frac{800\text{rpm}}{500\text{rpm}} = 24\text{V} \end{aligned} \tag{21}$$

Now, knowing the induced voltage, the armature current, I can be found using the equivalent circuit, knowing that the armature and load resistances are in series,

$$I = \frac{E}{R + R_L} = \frac{24\text{V}}{0.625\Omega + 5\Omega} = 4.267\text{A} \tag{22}$$

The terminal voltage, V is given by,

$$V = IR_L = 4.267\text{A}\bullet 5\Omega = 21.34\text{V} \tag{23}$$

The electrical (output) power is,

$$P_e = \frac{V^2}{R_L} = I^2 R_L = VI = 21.34\text{V}\bullet 4.267\text{A} = 91.06\text{W} \tag{24}$$

The (input) mechanical power P_m,

$$\begin{aligned} P_m &= P_{em} + P_{nl} = EI + P_{nl} = 24\text{V}\bullet 4.267\text{A} + 10\text{W} \\ &= 102.4\text{W} + 10\text{W} = 112.4\text{W} \end{aligned} \tag{25}$$

The efficiency is given by the output power divided by the input power,

$$\eta = \frac{P_e}{P_m} = \frac{91.06\text{W}}{112.4\text{W}} = 88.9\% \qquad (26)$$

III. Surface PM Machines: Basic Theory

The surface PM machine equivalent circuit model (see Fig. 4) has many similarities to the DC machine model. It consists of a series combination of an induced voltage source, stator inductance and stator reactance. The key differences are the inclusion of the inductance and the fact that the voltages and currents are now AC not DC and thus the need to take into account their phase angles by expressing them as phasors (shown as bold).

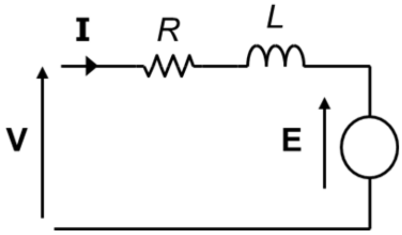

Fig. 4. Surface PM machine equivalent circuit.

A. Angular Frequency and Magnet Flux-Linkage

With AC machines there are two angular frequencies. Firstly there is the *mechanical* angular frequency, ω_m, which represents the mechanical rotation of the machine. This is used in the mechanical power calculation. Secondly is the *electrical* angular frequency, ω_e, which represents the frequency of the AC voltages and currents in the machine. This is used when finding the back-emf voltage and inductive reactance. The two frequencies are related by the number of pole-pairs in the machine, p which is half the number of poles in the machine P,

$$\omega_e = p\omega_m = \frac{P}{2}\omega_m \quad \text{[rad/s]} \qquad (27)$$

For example, consider a four-pole machine rotating at 1500rpm. The mechanical and electrical angular speeds are,

$$\begin{aligned} \omega_m &= \frac{1500\text{rpm} \bullet 2\pi}{60} = 157.1\text{rad/s} \\ \omega_e &= 2\text{pole-pairs} \bullet \omega_m = 314.2\text{rad/s} \end{aligned} \qquad (28)$$

The RMS phase induced voltage E is given by the product of the magnet flux linkage Λ_m and the electrical angular speed ω_e,

$$E = \Lambda_m \omega_e \quad \text{[V]} \qquad (29)$$

Note other common symbols for the magnet flux linkage are Ψ_m and λ_m.

Care should taken in applying the above equation as there are three aspects of uncertainity.

Firstly, the issue of RMS versus peak. In papers focussed on electric machine design, RMS values are often used to more easily compare the results with experimental measurements, while in papers focussed on electrical machine control, peak values are often used as this simplifies the control equations.

Secondly, line versus phase. If RMS values are used, there can be uncertainty about whether the back-emf constant is for the phase or line voltage.

Finally, whether the electrical or mechanical angular frequency is used in (29). Normally the electrical frequency is used. It is also possible that the back-emf constant be given in terms of V/rpm.

Due to the above possible aspects for confusion, it can sometimes be more precise to simply quote the back-emf at a given speed, e.g. for a six-pole machine the back-emf is 283Vrms (line) when the machine is spinning at 1,200 rpm.

For the above situation, the (phase rms) magnet flux linkage is given by,

$$\begin{aligned} \Lambda_m &= \frac{E}{\omega_e} = \frac{283\text{V}/\sqrt{3}}{3\text{pole-pairs}\left(\frac{1200\text{rpm} \bullet 2\pi}{60}\right)} \\ &= 0.4334\text{V/(rad/s)} \end{aligned} \qquad (30)$$

For a surface PM machine, the terminal voltage phasor **V** is given by,

$$\mathbf{V} = \mathbf{E} + \mathbf{I}(R + j\omega_e L) \quad \text{[V]} \qquad (31)$$

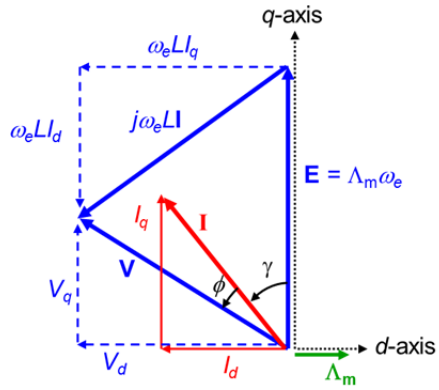

Fig. 5. Surface PM machine phasor diagram example.

B. D-Q Axis Equations and Model

The PM machine quantities are commonly expressed in the terms of the *d-q* axis components, where the magnet flux linkage is normally assumed to lie in the positive *d*-axis (see Figs. 5 and 6). This means the induced voltage E in the positive *q*-axis. Defining the current angle γ as the angle at which the stator current leads the *q*-axis, then,

$$\begin{aligned} I_q &= I\cos\gamma \\ I_d &= -I\sin\gamma \end{aligned} \qquad (32)$$

Thus if $\gamma = 0°$, then $I_q = I$ and $I_d = 0$. This is a common operating condition at low speeds for surface PM machines as it yields maximum torque for a given current (maximum torque per ampere). At higher speeds $\gamma > 0°$ is used in the so-called "field-weakening" region to allow operation at higher speeds with reduced torque.

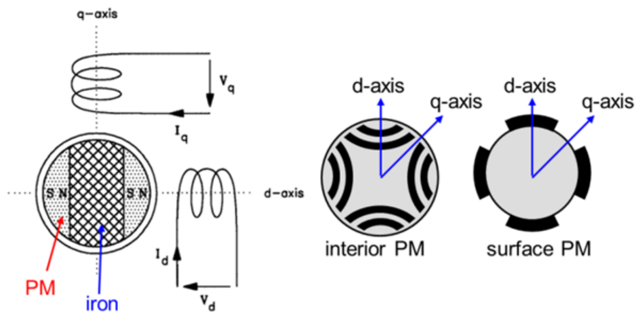

Fig. 6. Definition of *d-q* axes for PM machines.

Equation (31) can be expressed in terms of the d-axis and q-axis current components and inductances as,

$$V_q = \omega_e(\Lambda_m + I_d L_d) + I_q R \qquad V_d = -\omega_e I_q L_q + I_d R \tag{33}$$

The machine voltage equations can also be described in terms of the d- and q-axis flux linkages,

$$V_q = \omega_e \Lambda_d + I_q R \quad V_d = -\omega_e \Lambda_q + I_d R \quad \text{where} \quad \Lambda_d = \Lambda_m + I_d L_d \quad \Lambda_q = I_q L_q \tag{34}$$

These flux-linkage equations are convenient when discussing the field-weakening performance and control of the machine.

Equations (33-34) give the general form of the machine equations which are also applicable to interior PM machines where in general the inductances in the d- and q-axes are not equal $L_d \neq L_q$. For a surface PM machine $L_d = L_q = L$.

The magnitude of the stator (phase) current and voltage is given by,

$$I = \sqrt{I_d^2 + I_q^2} \qquad V = \sqrt{V_d^2 + V_q^2} \tag{35}$$

The machine voltage and currents can be expressed as phasors and are given by,

$$\mathbf{I} = \sqrt{I_d^2 + I_q^2}\angle \tan^{-1}\left(-\frac{I_d}{I_q}\right) \qquad \mathbf{V} = \sqrt{V_d^2 + V_q^2}\angle \tan^{-1}\left(-\frac{V_d}{V_q}\right) \tag{36}$$

Thus the power-factor angle ϕ of the current with respect to the voltage is given by,

$$\phi = \angle\mathbf{I} - \angle\mathbf{V} = \tan^{-1}\left(-\frac{I_d}{I_q}\right) - \tan^{-1}\left(-\frac{V_d}{V_q}\right) \tag{37}$$

The electrical power P_e for an m-phase machine is given by (where m is normally three),

$$P_e = mVI\cos\phi = 3VI\cos\phi \quad [\text{W}] \tag{38}$$

The electromagnetic torque T_{em} for a surface PM machine is related to the number of pole-pairs p, the magnet flux-linkage Λ_m and the q-axis current,

$$T_{em} = mp\Lambda_m I_q = mp\Lambda_m I\cos\gamma \quad [\text{Nm}] \tag{39}$$

The electromagnetic power P_{em} can be given in terms of the electromagnetic torque or else the power absorbed by the back-emf voltage source E,

$$P_{em} = T_{em}\omega_m = mp\Lambda_m I_q \omega_m = mEI_q \quad [\text{W}] \tag{40}$$

Like the DC machine, the relationship between the mechanical power P_m, electromagnetic power P_{em} and the no-load loss power P_{nl}, depends on whether the machine is motoring or generating, see (6). The same relationships for the torque, see (7), and efficiency, see (9), also apply.

C. Example #1: Motoring Case

Consider a star-connected surface PM machine with 48 poles, a magnet flux linkage of 0.0257 V/(rad/s), a stator inductance of 2.82mH, and a stator resistance of 0.524Ω. A current of 5Arms is applied at an angle of 30° leading the q-axis when the machine is spinning at 500rpm. The no-load loss at this speed is 30W. Find the stator voltage, power-factor, output torque and power and efficiency.

The mechanical and electrical angular speeds are,

$$\omega_m = \frac{500\text{rpm}\bullet 2\pi}{60} = 52.36\text{rad/s} \qquad \omega_e = p\omega_m = 24\text{pole-pairs}\bullet 52.36\text{rad/s} = 1257\text{rad/s} \tag{41}$$

The induced voltage is,

$$E = \Lambda_m\omega_e = 0.0257\text{V/(rad/s)}\bullet 1257\text{rad/s} = 32.31\text{V} \tag{42}$$

The terminal voltage V can be found by expressing E and I as phasors and solving using (31),

$$\begin{aligned}&\mathbf{E} = 32.31\angle 0°\text{V} \quad \text{and} \quad \mathbf{I} = 5\angle 30°\text{A}\\ &\mathbf{V} = \mathbf{E} + \mathbf{I}(R + j\omega L) = 32.31\angle 0°\text{V} + \\ &\quad 5\angle 30°\text{A}\bullet(0.524\Omega + j\bullet 1257\text{rad/s}\bullet 0.00282\text{H})\\ &= 30.64\angle 32.93°\text{V}\end{aligned} \tag{43}$$

A similar calculation can be performed using d-q quantities based on (32) and (33),

$$\begin{aligned}I_q &= I\cos\gamma = 5\bullet\cos 30° = 4.330\text{A}\\ I_d &= -I\sin\gamma = -5\bullet\sin 30° = -2.500\text{A}\end{aligned} \tag{44}$$

Then the d-q axis voltages can be found as,

$$\begin{aligned}V_q &= \omega_e(\Lambda_m + I_d L_d) + I_q R\\ &= 1257\text{rad/s}(0.0257 - 2.5\text{A}\bullet 0.00282\text{H}) + \\ &\quad 4.33\text{A}\bullet 0.524\Omega = 25.71\text{V}\\ V_d &= -\omega_e(I_q L_q) + I_d R\\ &= -1257\text{rad/s}(4.33\text{A}\bullet 0.00282\text{H}) - \\ &\quad 2.5\text{A}\bullet 0.524\Omega = -16.66\text{V}\end{aligned} \tag{45}$$

From which the magnitude and phase of V can be found as,

$$\begin{aligned}V &= \sqrt{V_q^2 + V_d^2} = \sqrt{(25.71\text{V})^2 + (-16.66\text{V})^2}\\ &= 30.63\text{V}\\ \angle V &= \tan^{-1}\left(-\frac{V_d}{V_q}\right) = \tan^{-1}\left(-\frac{-16.66\text{V}}{25.71\text{V}}\right) = 32.94°\end{aligned} \tag{46}$$

This matches the result shown earlier in (43). Though the d-q approach is clearly slower than (43), it has the advantage that it is also applicable to interior PM machines where the approach in (43) cannot be used.

The power-factor angle ϕ is given by the phase angle of the current with respect to the voltage,

$$\phi = \angle\mathbf{I} - \angle\mathbf{V} = 30° - (32.94°) = -2.94° \tag{47}$$

The input electrical power is then given by,

$$\begin{aligned}P_e &= 3VI\cos\phi = 3\bullet 30.63\text{V}\bullet 5\text{A}\bullet\cos(-2.94°)\\ &= 458.9\text{W}\end{aligned} \tag{48}$$

The electromagnetic torque,

$$\begin{aligned}T_{em} &= mp\Lambda_m I_q\\ &= 3\bullet 24\text{pole-pairs}\bullet 0.0257\text{V/(rad/s)}\bullet 4.33\text{A}\\ &= 8.012\text{Nm}\end{aligned} \tag{49}$$

The electromagnetic power,

$$P_{em} = T_{em}\omega_m = mEI_q = 3 \bullet 32.31\text{V} \bullet 4.33\text{A} = 419.7\text{W} \quad (50)$$

The mechanical power is,

$$P_m = P_{em} - P_{nl} = 419.7\text{W} - 30\text{W} = 389.7\text{W} \quad (51)$$

The efficiency is,

$$\eta = \frac{P_m}{P_e} = \frac{389.7\text{W}}{458.9\text{W}} = 84.9\% \quad (52)$$

D. Efficiency Maps

The above approach can be used to calculate the efficiency of the machine at regular intervals over the torque versus speed plane, allowing the calculation of an efficiency contour plot. The Matlab code for this is given in the appendix.

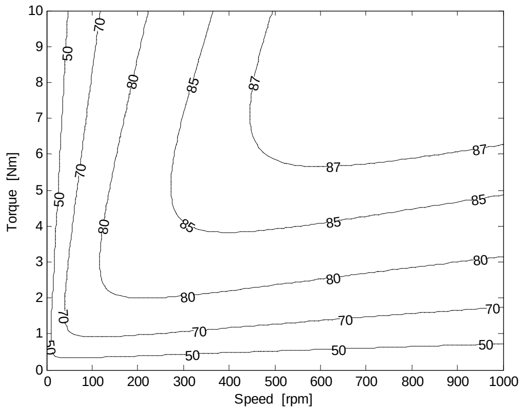

Fig. 7. Calculated motoring efficiency map for a surface PM machine.

E. Example #2: Generating Case

Consider the above surface PM machine acting as a generator with a three-phase resistive load of 6Ω per phase at the same speed.

The equivalent circuit consists of the induced voltage with the series combination of the stator inductance and the stator resistance and load resistance. Thus the stator current *I* is,

$$\mathbf{I} = \frac{\mathbf{E}}{(R + R_L) + j\omega L} = \frac{32.31\angle 0°\text{V}}{(0.524\Omega + 6\Omega) + j1257\text{rad/s} \bullet 0.00282\text{H}} = 4.352\angle -28.52°\text{A} \quad (53)$$

The output voltage *V* is given by,

$$\mathbf{V} = \mathbf{I}R_L = (4.352\angle -28.52°\text{A}) \bullet 6\Omega = 26.11\angle -28.52°\text{V} \quad (54)$$

The output electrical power is given by the following equation, where the power-factor angle ϕ is zero because the load is purely resistive,

$$P_e = 3VI\cos\phi = 3 \bullet 26.11\text{V} \bullet 4.352\text{A} \bullet 1 = 340.9\text{W} \quad (55)$$

The input electromagnetic power is given by the following, where γ is the angle between *E* and *I*,

$$P_{em} = 3EI\cos\gamma = 3 \bullet 32.31\text{V} \bullet 4.352\text{A} \bullet \cos(-28.52°) = 370.7\text{W} \quad (56)$$

The mechanical power is given by,

$$P_m = P_{em} + P_{nl} = 370.7\text{W} + 30\text{W} = 400.7\text{W} \quad (57)$$

The efficiency is given by,

$$\eta = \frac{P_e}{P_m} = \frac{340.9W}{400.7W} = 85.8\% \quad (58)$$

IV. CONSTANT TORQUE AND FIELD-WEAKENING OPERATION

Electric motor drives have three main modes of operation as shown in Fig. 8. The constant torque region extends from zero to rated (or knee) speed. In this region, the maximum torque is achieved by operating with rated current with a current angle to maximise the torque. The voltage (ignoring the resistive voltage drop) increases linearly with speed and reaches rated voltage at rated speed.

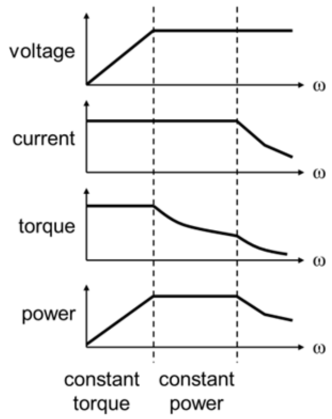

Fig. 8. Constant torque and constant power operating regions

The constant power (or field-weakening) region extends from rated speed to a higher speed limit. During this region, the output power is constant and so the output torque falls inversely with speed. The voltage and current remain at (or near) their rated values. The constant power speed range is the ratio of the maximum speed at which constant power can be maintained to the rated speed. Typical values for traction drives are 3:1 to 5:1.

Finally above the maximum constant power speed, the output power falls with increasing speed.

A. Constant Torque Operation

For a surface PM machine the output torque is only a function of the *q*-axis current. Thus for maximum torque for a given current (maximum torque per ampere, MTPA), the machine should be operated with $I_q = 0$. Consider a machine with a rated current I_0 and rated voltage V_0. In this case, the stator resistance will be neglected for simplicity.

At low speeds, maximum (usually called rated) torque T_0 is obtained by operating with rated current I_0 at a current angle $\gamma = 0°$, and hence $I = I_q = I_o$. This is sometimes referred to operating with rated current in the *q*-axis.

For the surface PM machine described in the previous section, and assuming a rated current I_0 of 5A, the maximum electromagnetic torque is,

$$T_{em} = mp\Lambda_m I_0 = 3 \bullet 24\text{pole-pairs} \bullet 0.0257\text{V/(rad/s)} \bullet 5\text{A} = 9.252\text{Nm} \quad (59)$$

The terminal voltage (neglecting resistance) is given by,

$$V = \sqrt{(\omega_e \Lambda_m + \omega_e L_d I_d)^2 + (\omega_e L_q I_q)^2} = \omega_e \sqrt{(\Lambda_m + L_d I_d)^2 + (L_q I_q)^2} \quad (60)$$

which is directly proportional to speed. The terminal voltage will reach the rated voltage when,

$$\omega_e = \frac{V_0}{\sqrt{(\Lambda_m + L_d I_d)^2 + (L_q I_q)^2}} \quad (61)$$

For the surface PM machine under maximum torque per ampere operation, this occurs at the so called, the knee speed, ω_{ek},

$$\omega_{ek} = \frac{V_0}{\sqrt{(\Lambda_m)^2 + (L_q I_0)^2}} \quad (62)$$

Assume a rated voltage V_0 of 30V, this would be,

$$\omega_{ek} = \frac{30\text{V}}{\sqrt{(0.0257\text{V/(rad/s)})^2 + (0.00282\text{H}\bullet 5\text{A})^2}} = 1023\text{rad/s} \equiv 407.2\text{rpm} \quad (63)$$

Thus this machine is capable of producing its rated torque of about 9.25 Nm at speeds of up to about 407 rpm (neglecting resistance). This is called the constant torque operation region.

At the knee speed, the electromagnetic output power is,

$$P_{em} = T_{em}\omega_m = 9.252\text{Nm}\bullet\frac{1023\text{rad/s}}{24\text{pole-pairs}} = 394.4\text{W} \quad (64)$$

B. Field-Weakening Operation

Once the terminal voltage reaches its maximum value V_0, this cannot be exceeded. Thus to operate the surface PM machine at higher speeds with the maximum output power, it can be shown that it is necessary to increase the current angle γ from 0° while keeping the current magnitude at rated current I_0. This increases the negative value of I_d which reduces the required terminal voltage at a given speed and thus allows the speed to be increased. This is called field-weakening where the machine operates above rated speed, but with an output torque which is less than rated torque.

While the *d*-axis current does not contribute to torque production in a surface PM machine, it does change the *d*-axis flux-linkage in the machine and hence the required terminal voltage.

Consider the example machine operating with rated current at a current angle γ = 30°. This is similar to the motoring example given above. In this case, from (49) the electromagnetic torque is 8.012 Nm which is less than rated torque. However the machine can now operate to a higher speed from (44) and (61),

$$\omega_e = \frac{V_0}{\sqrt{(\Lambda_m + L_d I_d)^2 + (L_q I_q)^2}} = \frac{30\text{V}}{\sqrt{\left(0.0257 - 0.00282\bullet 2.5\right)^2 + (0.00282\bullet 4.33)^2}} = 1346\text{rad/s} \equiv 535.5\text{rpm} \quad (65)$$

which is about 30% higher than rated speed.

The electromagnetic output power at this speed is,

$$P_{em} = T_{em}\omega_m = 8.012\text{Nm}\bullet\frac{1346\text{rad/s}}{24\text{pole-pairs}} = 449.3\text{W} \quad (66)$$

which is actually about 10% higher than the output power at the knee speed. It is normal that the maximum output power at the start of the field-weakening region actually increases, even though the output torque is decreasing. This gives rise to the term the constant power speed range (CPSR) as the range of speeds over which the maximum output power is greater than the power at the knee speed.

A special case is where the current angle γ = 90°. In this case, $I_d = -I_0$ and $I_q = 0$. As the *q*-axis current is zero, thus the output torque is zero. The maximum operating speed under this circumstance is,

$$\omega_e = \frac{V_0}{\sqrt{(\Lambda_m + L_d I_0)^2}} = \frac{30\text{V}}{\sqrt{\left(0.0257 - 0.00282\bullet 5\right)^2}} = 2586\text{rad/s} \equiv 1029\text{rpm} \quad (67)$$

which is about 2.5 times rated speed. This represents the maximum operating speed for the example motor with the given values of rated voltage and current. This speed can be increased if either the voltage or current limits are increased. Normally current limits are associated with thermal limits of the motor, power electronics or power source, and so often may be exceeded for short periods during transient operation.

Using the above technique for a range of current angles between 0° and 90°, it is possible to map out the field-weakening torque versus speed characteristic of the machine.

It can be shown that the optimum field-weakening performance for a surface (and interior) PM machine is achieved when the following relationship occurs between the magnet flux-linkage, *d*-axis inductance and rated current,

$$\Lambda_m = L_d I_0 \quad (68)$$

Under these circumstances, ideally, the machine can operate to any speed [1,2]. In practice, maximum operating inverter frequencies and rotor mechanical limitations will affect the highest possible operating speed.

For most surface and interior PM motors, the magnet flux-linkage is greater than the product of *d*-axis inductance and rated current [2]. These machines have a maximum operating speed given by (67). For these machines, maximum torque in the field-weakening region is obtained by operating with maximum current at the appropriate current angle to give rated voltage.

For machines where the magnet flux-linkage is less than or equal to product of *d*-axis inductance and rated current, they have no electrically limited maximum speed. In the field-weakening region, at higher speeds, maximum torque is sometimes obtained by using less than rated current [2].

V. Interior PM Machines

A. Interior PM Machine Analysis

The analysis of interior PM machines is similar to that described for surface PM machines however a key difference is that in these machines, the *d*-axis and *q*-axis inductances are no longer equal. This means that is not possible to use Eqn. (31) to analyse the machine. Instead the *d*- and *q*-axis equations or phasor diagram must be used.

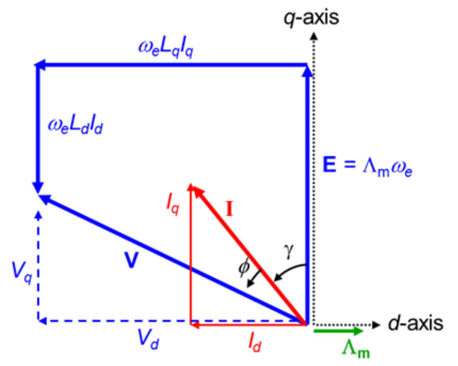

Fig. 9. Interior PM machine phasor diagram example.

The ratio of the q-axis and d-axis inductances is known as the saliency ratio ξ,

$$\xi = \frac{L_q}{L_d} \tag{69}$$

For most interior PM machines $L_q > L_d$ and hence $\xi > 1$. There are some unusual interior PM designs where $L_q < L_d$ and these are sometimes referred to as "inverse-saliency" designs.

For surface and interior PM machines the *d*-axis is normally defined as the axis which the magnet flux-linkage lies in. This is normally the low inductance axis. This is a possible cause of confusion as in synchronous reluctance machines the *d*-axis is normally defined as the high inductance axis [3]. In some synchronous reluctance machine designs, a small amount of PM material is added to improve their performance to form what is sometimes referred to as a permanent-magnet assisted synchronous reluctance. Such machines normally follow the synchronous reluctance convention of having the most inductive axis being the *d*-axis.

The difference between the *d*- and *q*-axis inductances mean that the machine torque equation now has a second "reluctance" term as shown,

$$\begin{aligned} T_{em} &= mp\left(\Lambda_m I_q + (L_d - L_q) I_d I_q\right) \\ &= mp\left(\Lambda_m I\cos\gamma - 0.5(L_d - L_q) I^2 \sin 2\gamma\right) \end{aligned} \tag{70}$$

This reluctance term means that the maximum torque at low speeds for a given current no longer occurs at a current angle $\gamma = 0°$. This is because of the sin 2γ term the maximum of the reluctance torque term occurs at $\gamma = 45°$, and thus the maximum of the sum of the two torque components would thus ideally occur with a current angle between 0° and 45° depending on the ratio of the amplitudes of the two torque components. This is shown in Fig. 10.

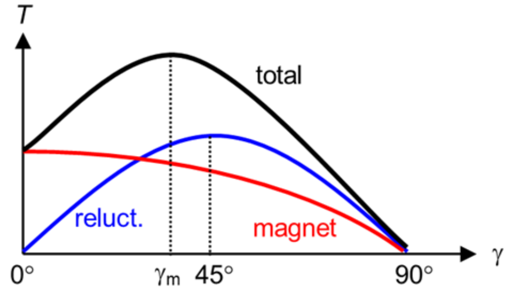

Fig. 10. Torque versus current angle for the magnet, reluctance and total torque.

For a given current magnitude *I*, the current angle γ_m corresponding to maximum torque can be obtained by differentiating (70) with respect to current angle,

$$\sin\gamma_m = \frac{-\Lambda_m + \sqrt{\Lambda_m^2 + 8\left(L_q - L_d\right)^2 I^2}}{4\left(L_q - L_d\right) I} \tag{71}$$

which is dependent on the three interior PM machine equivalent circuit parameters: Λ_m, L_q and L_d, as well as the magnitude of the current *I*.

B. Worked Example

Consider the same machine as in Section IV except this time assume $L_q = 2L_d$ where $L_d = 2.82$ mH. Resistance will again be neglected and the same assumptions of a rated current $I_0 = 5$A and a rated voltage $V_0 = 30$V will be used.

The maximum-torque-per-ampere current angle at rated current is found from (71) noting that in this special case $L_q - L_d = L_d$,

$$\sin\gamma_m = \frac{-0.0257 + \sqrt{0.0257^2 + 8\left(0.00282\right)^2 5^2}}{4\left(0.00282\right)5} \tag{72}$$

where $\gamma_m = 22.68°$ which, as expected, is between 0° (a surface PM machine) and 45° (a pure reluctance machine). From this,

$$\begin{aligned} I_q &= I\cos\gamma_m = 5\text{A}\bullet\cos(22.68°) = 4.628\text{A} \\ I_d &= -I\sin\gamma_m = -5\text{A}\bullet\sin(22.68°) = -1.892\text{A} \end{aligned} \tag{73}$$

The electromagnetic torque is given by,

$$\begin{aligned} T_{em} &= mp\left(\Lambda_m I_q + (L_d - L_q) I_d I_q\right) \\ &= 8.537\text{Nm} + 1.806\text{Nm} = 10.34\text{Nm} \end{aligned} \tag{74}$$

which shows that while the interior PM machine's PM torque is less than that of the surface PM machine (in Eqn (59) found to be 9.252Nm) however the additional reluctance torque component produces a total torque component which is greater than that of the surface PM machine.

The rated speed is given by,

$$\begin{aligned} \omega_e &= \frac{V_0}{\sqrt{(\Lambda_m + L_d I_d)^2 + (L_q I_q)^2}} \\ &= \frac{30\text{V}}{\sqrt{(\Lambda_m - 0.00282\bullet1.892)^2 + (0.00564\bullet4.628)^2}} \\ &= 906.1\text{rad/s} \equiv 360.5\text{rpm} \end{aligned} \tag{75}$$

The electromagnetic output power at this speed is 390.4W. Thus while the rated speed of the example interior PM is lower than the example surface PM machine (407rpm), it has a similar rated power to the surface PM machine (394W).

C. Saturation and Cross-Saturation

The above analysis has assumed that the inductances in the machine are constant and independent of current. Generally the *q*-axis inductance of interior PM machines shows significant saturation. To a first approximation saturation effects can be modelled by considering each axis inductance to be a function of the current in the same axis, e.g. $L_q(I_q)$ and $L_d(I_d)$.

When solving the equivalent circuit, if I_d and I_q are known then the appropriate values of L_d and L_q can be used. If I_d and I_q are solved for using the equivalent circuit, then an iterative solution may be required.

Cross-saturation effects can also be considered where L_d and L_q are each assumed to be functions of both I_d and I_q.

The variation of L_d and L_q with I_d and I_q is usually obtained from finite-element analysis or experimental testing. An example saturation curve is shown in Fig. 11.

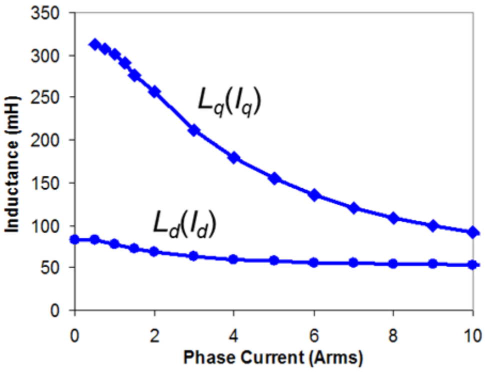

Fig. 11. Example saturation curves for an interior PM machine.

VI. Conclusions

The electrical equivalent circuit forms a key part to electric machine analysis, design and control. This chapter described the equivalent circuits and steady-state analysis of dc machines, surface permanent magnet machines and interior permanent magnet machines. It also includes discussion of field-weakening control and saturation.

References

[1] R. F. Schiferl and T. A. Lipo, "Power capability of salient pole permanent magnet synchronous motors in variable speed drive applications," *Industry Applications, IEEE Transactions on,* vol. 26, pp. 115-123, 1990.

[2] W. L. Soong and T. J. E. Miller, "Field-weakening performance of brushless synchronous AC motor drives," *Electric Power Applications, IEE Proceedings -,* vol. 141, pp. 331-340, 1994.

[3] M. Ferrari, N. Bianchi, A. Doria, and E. Fornasiero, "Design of synchronous reluctance motor for hybrid electric vehicles," in *Electric Machines & Drives Conference (IEMDC), 2013 IEEE International*, 2013, pp. 1058-1065.

Appendix : Matlab Code for Efficiency Map

```
% draws an efficiency map for an AC machine
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 1 : Create Torque and Speed Matrices
clear;          % clear all variables
NPoints = 100;  % array size for efficiency calculation
Tmax =  10;     % rated torque in Nm
Nmax = 1000;    % rated speed in rpm
Wmax = Nmax*2*pi/60;
T_vector = linspace(0,Tmax,NPoints);  % vector of torque values
n_vector = linspace(0,Nmax,NPoints);  % vector of mech speed rpm
wm_vector = n_vector*2*pi/60;         % vector of mech speed rad/s
[wm,T] = meshgrid(wm_vector+Wmax/1000,T_vector+Tmax/1000);
   % the terms Wmax/1000 and Tmax/1000 avoid divide-by-zero errors
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2 : Calculate Efficiency Map
R = 0.524;     % stator phase resistance, ohms
p = 24;        % pole-pairs
Psim = 0.0257; % back-emf constant, rms phase volts per elec rad/s
Tloss = -7.68E-06 .* wm.^2 + 5.10E-03 .* wm + 2.73E-01;
   % loss torque equation determined from no-load test results
Tem = T + Tloss; % electromagnetic torque = load + loss torque
we = p.*wm;     % electrical rad/s
E = Psim .*we;      % rms phase back-emf voltage
I = (Tem .* wm)./ (3 .* E);
   % rms phase current, assume I is in phase with E, from 3EI = Tw
eff= (T .* wm) ./ (Tem .* wm + 3 .* I.^2 .* R );
   % efficiency calc’n including no-load loss and stator I^2 R
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 3 : Plot Out Efficiency Map
[c,h] = contour(n_vector,T_vector,eff*100,[50 70 80 85 87],'k');
clabel(c,h);     % show contour labels
xlabel('Speed  [rpm]');
ylabel('Torque  [Nm]');
```

A Word for Today

“Jesus answered, “I am the way and the truth and the life. No one comes to the Father except through me.”

John 14:6 (NIV)