We propose a mechanism whereby a random alternation of two dynamics each leading to a different homogeneous state can lead to complex ordered structures. The proposed general formalism, based on the ideas of so-called paradoxical games, is illustrated via numerical simulations of particular examples. The relevance of the present study to other situations that lead to pattern formation, such as reaction-diffusion systems, is noted.

Keywords: Random switching; global switching; Parrondo's paradox; pattern formation.

1. Introduction

It has recently been established that the outcome of two alternating stochastic dynamics can differ significantly from that of each separate dynamics. The phenomenon has been illustrated with two losing gambling games that turn into a winning game when they are alternated either randomly or periodically [1,2], and has been extended to other situations such as non-Markovian [3] and cooperative [4] games. This counterintuitive phenomenon, frequently referred to as Parrondo’s paradox, was originally inspired by Brownian flashing ratchets [5–7], where a Brownian particle exhibits systematic motion in one direction when a periodic potential is alternately switched on and off.

The main message of the paradox is that the alternation of stochastic dynamics can yield new and counterintuitive phenomena. Davies [8] has also pointed out that the paradox provides a simple mechanism to create order (systematic motion or systematic gain) from fluctuations, and that a mechanism of this kind is probably needed for an explanation of the origin of life.
One of the interesting questions prompted by the paradox is whether order and complexity can arise if two dynamics, each of which separately drives the system to a disordered state or to a homogeneous state, are alternated. Here we show that a wide class of spatially extended systems can exhibit this unexpected behavior. These systems are based on the Swift-Hohenberg equation [9], which has been extensively used to model the appearance of spatial patterns in convection, but the mechanism underlying this “paradoxical” behavior can also apply to other spatially extended systems such as reaction-diffusion equations. Moreover, the mechanism presented in this paper helps to understand the role of global noise in spatially extended systems. It is known that local (site-dependent) fluctuations may induce pattern formation [10,11]. On the other hand, global periodic forcing as a method of generating spatial structures has been extensively studied since the seminal work of Faraday [12] and continues to lead to striking new results, such as the recently found oscillons or particle-like spatial excitations [13,14]. However, the use of global noisy forcing to obtain spatial patterns is, to our knowledge, uncharted territory both from the theoretical and experimental points of view. The present study sheds light on this issue and gives explicit solutions for a large family of systems.

The paper is organized as follows. In Sec. 2 we state the problem of random alternating dynamics for scalar fields obeying overdamped evolution equations. We also elucidate the general conditions to obtain spatial structures by random switching between dynamics. Section 3 is devoted to the analysis of a particular example. In Sec. 4 we present the main conclusions and point to some ideas for future work.

2. General Formalism
Consider the following Langevin dynamics for a scalar field $\varphi$:

$$\dot{\varphi}(\mathbf{r}, t) = -V'(\varphi(\mathbf{r}, t)) + L\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t),$$

(1)

where $\varphi(\mathbf{r}, t)$ depends on both space, $\mathbf{r}$, and time, $t$. The field could, for example, represent the concentration of a chemical species at a given spatial position and time. The temporal evolution of $\varphi(\mathbf{r}, t)$ is driven by a local force that can be derived from a local potential, $V(\varphi)$, by its coupling with the field at other locations through the operator $L$, and by thermal fluctuations $\xi(\mathbf{r}, t)$, taken to be Gaussian white noise with zero mean and correlation function

$$\langle \xi(\mathbf{r}, t)\xi(\mathbf{r}', t') \rangle = \sigma^2 \delta(\mathbf{r} - \mathbf{r}')\delta(t - t').$$

(2)

For a system such as (1), there are two required features for pattern formation: local bistability, that is, the local potential must show two stable equilibrium points, and a morphological instability [15]. A paradigmatic example is the Swift-Hohenberg equation [9], a model that describes Rayleigh-Benard convection near the threshold for the appearance of convective rolls. The Swift-Hohenberg equation is intended as a simplified form of the full Navier-Stokes equation for fluid flow that avoids the intractable complexity of the full equation, and yet is capable of generating some of the patterns observed in convection. A brief qualitative analysis of this model is useful to clarify the features of the systems exhibiting patterns induced by the alternation of dynamics. Equation (1) represents the Swift-Hohenberg model.
when the local potential and the coupling term are respectively chosen as
\[ V(\varphi) = \frac{1}{4} \varphi^4 - \frac{a}{3} \varphi^3 - \frac{b}{2} \varphi^2, \quad (3) \]
\[ L = -(1 + \nabla^2)^2. \quad (4) \]

We use this coupling operator throughout this paper. The morphological instabilities leading to pattern formation are identified by determining a uniform solution of the (noiseless) evolution equation (1) and linearizing about this solution. With the local potential (3) we have \( V'(0) = 0 \) and hence the uniform stationary solution is \( \varphi = 0 \). Linearization about this field gives the evolution equation
\[ \dot{\varphi}(\mathbf{r}, t) = -[V''(0) + (1 + \nabla^2)^2] \varphi(\mathbf{r}, t) \quad (5) \]
or, performing a spatial Fourier transform (denoted by a hat),
\[ \dot{\hat{\varphi}}(\mathbf{k}, t) = -[V''(0) + (1 - |\mathbf{k}|^2)^2] \hat{\varphi}(\mathbf{k}, t). \quad (6) \]

The least stable modes thus correspond to \( |\mathbf{k}| = 1 \), and a morphological instability occurs if \( V''(0) < 0 \). Regarding the local potential, we distinguish different cases according to the values of the parameters \( a \) and \( b \). If \( a \) and \( b \) are such that \( V(\varphi) \) is monostable, no spatial structures appear. On the other hand, if \( a \) and \( b \) are such that the local potential is bistable, patterns are obtained. The shape of the patterns depends on the symmetry of \( V(\varphi) \): if \( V(\varphi) = V(-\varphi) \) the system develops convection rolls, but if Eq. (1) does not possess the symmetry \( \varphi \leftrightarrow -\varphi \) then spot-shaped patterns are observed, usually arranged in a hexagonal lattice.

Consider now a random global alternation of the local potential in (1), i.e.,
\[ \dot{\varphi}(\mathbf{r}, t) = -\Lambda(t)V'_1(\varphi(\mathbf{r}, t)) - (1 - \Lambda(t))V'_2(\varphi(\mathbf{r}, t)) \]
\[ + L\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t), \quad \text{where } \Lambda(t) \text{ is a dichotomous random variable that assumes the values 0 and 1 with equal probability, and with correlation function} \]
\[ \frac{\langle \Lambda(t)\Lambda(t') \rangle - \langle \Lambda(t) \rangle \langle \Lambda(t') \rangle}{\langle \Lambda^2(t) \rangle - \langle \Lambda(t) \rangle^2} = \exp(-|t - t'|/\tau), \quad \text{(8)} \]

\( \tau \) being the correlation time of the switching process. We further require that the local potentials \( V_{1,2}(\varphi) \) each have only one stable equilibrium point. Therefore, at a given time, all the sites in the system are driven either by \( V_1(\varphi) \) or by \( V_2(\varphi) \). Notice that each of these dynamics if taken separately cannot generate any patterns because both are monostable.

In terms of the potentials
\[ V_{\pm}(\varphi) = \frac{V_1(\varphi) \pm V_2(\varphi)}{2}, \quad \text{(9)} \]
and a new dichotomous variable \( \mu(t) = \pm 1 \), Eq. (7) can be rewritten as
\[ \dot{\varphi}(\mathbf{r}, t) = -V'_+(\varphi(\mathbf{r}, t)) - \mu(t)V'_-(\varphi(\mathbf{r}, t)) + L\varphi(\mathbf{r}, t) + \xi(\mathbf{r}, t). \quad \text{(10)} \]

Note that \( \langle \mu(t)\mu(t') \rangle = \exp(-|t - t'|/\tau). \)
The behavior of the system as a function of the switching correlation time is as follows. If \( \tau \) is large, the system alternates between two homogeneous states, one associated with \( V_1(\varphi) \) and the other with \( V_2(\varphi) \). On the other hand, if \( \tau \to 0 \) but the noise intensity remains finite, the fluctuations are negligible and the noise can simply be approximated by its mean value (only if the intensity of the noise goes to infinity do fluctuations continue to play a role in the vanishing correlation time limit; in this case it is well known [16] that dichotomous noise tends to a white Gaussian noise). The mean value of \( \mu(t) \) is zero, and the dynamics in this limit is therefore driven by the effective potential \( V_+^e(\varphi) \). The stability analysis leading to Eq. (6) must now be slightly modified, however, because \( V_+^e(0) \) is in general not zero, that is, if there is a uniform solution it may not occur at \( \varphi = 0 \). One must seek a uniform solution \( \varphi = \tilde{\varphi} + \Delta \varphi \) which satisfies the stationarity condition
\[
V_+^e(\tilde{\varphi} + \Delta \varphi) + \Delta \varphi = 0 \tag{11}
\]
where the last term arises from the 1 in the coupling operator L. Setting \( \varphi = \tilde{\varphi} + \Delta \varphi \), the stability diagnostic equation (6) must now be modified to
\[
\Delta \dot{\varphi}(k, t) = - \left[ V_+''(\tilde{\varphi}) + (1 - |k|^2) \right] \Delta \varphi(k, t), \tag{12}
\]
and a morphological instability sets in if \( V_+''(\tilde{\varphi}) < 0 \).

According to the above discussion the potentials \( V_{1,2}(x) \) must fulfill the following conditions to produce patterns by random alternation:

- **C1.** \( V_1'(\tilde{\varphi}_1) + \tilde{\varphi}_1 = 0 \) and \( V_2'(\tilde{\varphi}_2) + \tilde{\varphi}_2 = 0 \) each have a single stable solution \( (V_1''(\tilde{\varphi}_1) > 0, V_2''(\tilde{\varphi}_2) > 0) \)

- **C2.** \( V_+'(\tilde{\varphi}) + \tilde{\varphi} = 0 \) has at least one unstable solution \( (V_+''(\tilde{\varphi}) < 0) \)

We expect pattern formation when there is sufficiently rapid switching between \( V_1 \) and \( V_2 \). Moreover, note that in the limit \( \tau \to 0 \) the temporal dependence disappears from the local dynamics. Therefore, stationary patterns should be obtained as a result of rapid random alternation.

It is easy to demonstrate that conditions (C1–C2) can not be fulfilled if both local potentials are quadratic; we need at least one of the local potentials to be at least quartic. In general, one of the two potentials has to be a non-convex function, i.e., the second derivative can not have the same sign for all \( \varphi \). Consequently, non-linearity is a necessary ingredient for the proposed pattern formation mechanism.

### 3. A Particular Case: Numerical Simulations

Consider the following particular choice of potentials satisfying C1 and C2:

\[
V_1(\varphi) = A \left( \frac{\varphi^4}{4} + \frac{\varphi^3}{3} - \frac{\varphi^2}{2} - \varphi \right), \tag{13}
\]
\[
V_2(\varphi) = B \left( \frac{\varphi^4}{4} - \frac{\varphi^3}{3} - \frac{\varphi^2}{2} + \varphi \right),
\]
where \( A \) and \( B \) are positive constants. For this case
\[
V_+(\varphi) = \frac{1}{2} \left( \alpha \frac{\varphi^4}{4} + \beta \frac{\varphi^3}{3} - \alpha \frac{\varphi^2}{2} - \beta \varphi \right), \tag{14}
\]
where $\alpha = A + B$ and $\beta = A - B$. Depending on the values of the parameters $A$ and $B$, we can distinguish two cases. If $A = B$, $V_+(\varphi)$ is proportional to the even part of the local potentials. Following the parallels with the Swift-Hohenberg model discussed in the previous section, we expect rolls if the random alternation is sufficiently rapid. On the other hand, if $A \neq B$ the effective local potential $V_+(\varphi)$ is not an even function, but it does have two stable states. In this case we expect spots (cf. Eqs. (14) and (3)).

We perform numerical simulations of Eq. (7) on a $128 \times 128$ lattice using the local potentials as defined in (13). The values of the relevant parameters are $\Delta t = 10^{-3}$, $\Delta x = \Delta y = 0.5$, $L_x = L_y = 64$, $\sigma = 10^{-2}$ and we implement periodic boundary conditions. The most unstable modes for the coupling operator used here are $|k^*| \simeq 1$ [17]; hence the wavelength of the pattern will be $\lambda = 2\pi/|k^*| \simeq 2\pi$ and the aspect ratio $L/\lambda \sim 10$. Note that the fluctuations must be sufficiently small not to swamp the potential barrier in $V_+$. In our simulations we take either the initial field to be random according to a Gaussian distribution (in which case the additive fluctuations can actually be omitted entirely), or we start from an arbitrary initial condition (e.g. all points equilibrated with $V_1$), in which case the (small) fluctuations will distribute the field in any case. Note that the role of the additive noise here is different from other systems, where noise terms do have a direct influence on the pattern selection mechanism (for an example in the Swift-Hohenberg context see e.g. [18]).

We focus the numerical simulations on the cases $A = B = 1$ (a) and $A = 1$, $B = 2$ (b). We expect transitions from alternating homogeneous states to rolls in the former and to spots in the latter as we decrease the switching correlation time.

The transition from alternating homogeneous states to pattern formation is clearly observed in the dependence of $\langle S(k) \rangle_{\theta}$, the spatial structure function averaged over angles, on the random switching correlation time $\tau$. The structure function is defined as usual, $S(k) = |\hat{\varphi}^2(k,t)|$, and its angular average as the average over $\theta$ when $k$ is expressed in polar coordinates. The resulting average is time-independent, that is, the resulting patterns are stationary. Figure 1 shows $\langle S(k) \rangle_{\theta}$ for case (a) for two different correlation times, $\tau = 1$ and $\tau = 10^{-2}$. The flat spectrum for $\tau = 1$ indicates that the system exhibits no spatial structure; the system is in fact simply alternating between homogeneous states. On the other hand, as evidenced by the peak at the most unstable mode that appears in the structure function for the case $\tau = 10^{-2}$, by sufficiently increasing the flipping rate we obtain an ordered structure, i.e., a pattern. Figure 2 shows the same quantity for case (b). Here again one obtains a pattern by randomly alternating sufficiently quickly between two potentials each of which leads to a homogeneous state.

Figure 3 shows density plots of the field for $\tau = 10^{-2}$ for cases (a) and (b). As expected, (a) leads to a roll-shaped pattern, while case (b) reproduces a spot-shaped pattern. The symmetry of the patterns can be elucidated by analyzing their structure functions $S(k)$. This quantity is plotted in Fig. 4 by means of a density plot on a logarithmic scale, revealing the circular symmetry of the rolls (isotropy), case (a), and the hexagonal symmetry of the spot-like pattern (anisotropy), case (b).
Fig. 1. Structure functions $S(k)$ averaged over angles for case (a) described in the text for two different values of the correlation time of the random switching. Note that no structure appears when alternation between dynamics is slow, as evident from the flat spectrum (circles). Fast switching leads to pattern formation with a well-defined spatial periodicity (squares).

Fig. 2. Angular average of the structure function $S(k)$ vs $k$ for case (b) described in the text for two different correlation times, $\tau = 1$ (circles) and $\tau = 10^{-2}$ (squares). The appearance of a peak for the mode $k \approx 1$ upon decreasing the random switching time clearly indicates the emergence of a regular structure.
Fig. 3. Density plot of the field for cases (a) and (b) for $\tau = 10^{-2}$. Convection rolls develop in case (a). Hexagonally arranged spot-like patterns appear in case (b). The symmetry of the effective local potential $V_\epsilon(\phi)$ determines the structure.

Fig. 4. Density plot of the logarithm of the structure functions associated with the structures of Fig. 3. A clear circular symmetry is obtained for case (a). In case (b) the spectrum indicates a hexagonal pattern.

4. Conclusions

We have presented a broad class of spatially extended systems where the alternation of two dynamics induces patterns whereas each separate dynamics drives the system to a homogeneous state. In this way we have partially answered the question of how complexity can arise from the alternation of non-complex dynamics. Moreover, we have demonstrated that global noisy forcing helps to develop ordered spatial structures.

The types of patterns that can be obtained depends on the choice of the local potentials or, more precisely, on their symmetries. For the particular case of a bistable effective potential, convection-rolls are obtained if $V_\epsilon(\phi)$ is an even function, whereas spot-shaped structures appear otherwise. This scenario is in agreement with the known phenomenology of the Swift-Hohenberg model.
If switching between dynamics is periodic rather than random, interesting resonance phenomena between the switching rate and the relaxation rate from nonequilibrium states are observed [14, 19]. For example, localized periodic excitations (oscillons) can occur. Multi-phase patterns have attracted a great deal of recent attention in cases where a periodic forcing is applied [20, 21]. These patterns raise interesting questions related to the possibility of creating multi-phase patterns in a system that simply alternates between two potentials each of which is associated with a homogeneous state.

It is worth noting that the mechanism we have presented is quite general and does not depend on the particular choice of the local potentials and/or the coupling term: other families of local potentials combined with coupling terms incorporating morphological instabilities can also develop spatial and/or temporal structures. For instance, we expect this mechanism to be relevant in reaction-diffusion systems [22].

Acknowledgments

We thank R. Kawai for fruitful discussions. This work was supported in part by the National Science Foundation under grant No. PHY-9970699, by DGES-Spain Grant PB-97-0076, by the New Del Amo Program, and by MECD-Spain Grant EX2001-02880680.

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